

ON THE NEWTON STRATIFICATION OF A SHIMURA CURVE OF HODGE TYPE: THE CASE OF CORESTRICTION

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ABSTRACT. This paper studies the Newton stratification of a certain Shimura curve of Hodge type in characteristic p . The main results are the determination of the Newton polygons and a mass formula on the cardinality of the supersingular points. The underlying technical results are a direct-tensor decomposition of the associated filtered Dieudonné module to a universal abelian scheme over the formal neighborhood of a characteristic p closed point of the integral canonical model of the Shimura curve, and the construction of a so-called weak Hasse-Witt pair by using p -adic Hodge theory and relative p -adic Hodge theory.

1. INTRODUCTION

The paper is to give two basic results on the Newton stratification of a certain Shimura curve of Hodge type in characteristic p : The first one is the determination of all possible Newton polygons; The second one is a mass formula for *the* nongeneric Newton strata. This work continues our previous one (see §2-5 [34]), and the principal aim here is to resolve Conjecture 1.3 in [34] for a Shimura curve of Hodge type, which arises from a generalization of the construction of D. Mumford (§4 [27]). In [34] (see also [23]), we studied a Shimura curve of PEL type: Deligne-Shimura's modèle étrange ([6]). The corresponding results can be relatively easily derived, mainly because of the existence of many endomorphisms in the generic fiber of a universal family, which yields a useful direct sum decomposition of the associated universal filtered Dieudonné module. A simple transplant of this method to the current case does not work any more: the generic endomorphism algebra is so small that one obtains no interesting direct decomposition on H^1 . Indeed, one of highlights of the example of Mumford (see loc. cit.) is that the endomorphism algebra is generically trivial. One of main contributions in this paper is to exhibit a formal tensor decomposition of the universal filtered Dieudonné module due to the presence of many Hodge cycles in the generic fiber. The decomposition result shows an intimate relation between the associated p -divisible groups and Drinfel'd \mathcal{O}_p -divisible modules, which seems to deserve a further understanding in the future. It contains implications on the Newton stratification of such a Shimura curve in characteristic p . The main tool is the p -adic Hodge theory ([8], [9], [11]-[13], [3]-[4], [17], [1]). The geometry of the Newton stratification of a Shimura variety in char p is rich. For a Shimura variety of PEL type, especially a higher dimensional one, the results are extensive,

This work is supported by the SFB/TR 45 'Periods, Moduli Spaces and Arithmetic of Algebraic Varieties' of the DFG.

among which we mention only two recent papers [39], [37] and refer the reader to the references therein for more literature. We hope that some techniques of this work could also be used for a Shimura variety of Hodge type.

Let F be a totally real field of degree d , whose ring of integers is denoted by \mathcal{O} , and D a quaternion division algebra over F , which is split only at one real place of F . The corestriction $\text{Cor}_{F|\mathbb{Q}} D$ is a central simple \mathbb{Q} -algebra. Generalizing a construction due to D. Mumford in loc. cit., one is able to associate $\text{Cor}_{F|\mathbb{Q}} D$ with a Shimura curve of Hodge type. Let p be a rational prime satisfying the following

Assumption 1.1. $p \geq 3$ and does not divide the discriminants of F and D .

After the work of Kisin [18], one is able to define the integral canonical model of the Shimura curve over any prime \mathfrak{p} of F over p ¹ and also a universal abelian scheme over the integral model, which is defined over $\mathcal{O}_{(\mathfrak{p})}$. Fix such a universal abelian scheme, and denote its completion at \mathfrak{p} by $f : X \rightarrow M$ and the reduction modulo \mathfrak{p} by $f_0 : X_0 \rightarrow M_0$. Put $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$ and fix an algebraic closure \bar{k} of a finite field k in characteristic p . Our first main result is the following

Theorem 1.2. *Notation and assumption on p as above. Then there are two Newton polygons in $M_0(\bar{k})$. Precisely it is either $\{2^{d+\epsilon(D)} \times \frac{1}{2}\}$ (i.e. supersingular) or*

$$\{2^{d-r+\epsilon(D)} \times 0, \dots, 2^{d-r+\epsilon(D)} \cdot \binom{r}{i} \times \frac{i}{r}, \dots, 2^{d-r+\epsilon(D)} \times 1\}.$$

Here $\epsilon(D)$ is either 0 or 1, depending on D only.

By the result, there is a unique nongeneric Newton strata: the supersingular locus. Building on the techniques in the proof of Theorem 1.2, we are able to show a *formal* tensor decomposition for the universal filtered Dieudonné module attached to f (see also Remark 4.14):

Theorem 1.3. *Let $x_0 \in M_0(k)$ and \hat{M}_{x_0} be the completion of M at x_0 . Then one has a natural tensor decomposition in the category $MF_{[0,1]}^{\nabla}(\hat{M}_{x_0})$ of the restriction to \hat{M}_{x_0} of the universal filtered Dieudonné module attached to f :*

$$(H_{dR}^1, F_{\text{hod}}, \nabla^{GM}, \phi)|_{\hat{M}_{x_0}} \cong \{[\otimes_{i=0}^{r-1} (\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \nabla_{\mathcal{N}_i}), \phi_{\text{ten}}] \otimes (M_{A_2}, \text{Fil}_{A_2}^1, d, \phi_{A_2})\}^{\oplus 2^{\epsilon(D)}},$$

where $\{(\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \nabla_{\mathcal{N}_i})\}_{0 \leq i \leq r-1}$ are all eigen components of the universal filtered Dieudonné module of the versal deformation of a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$ -divisible module and ϕ_{ten} is the tensor product of the ϕ_i s on eigen components, and $(M_{A_2}, \text{Fil}_{A_2}^1, d, \phi_{A_2})$ is a constant unit crystal.

With the help of the formal tensor decomposition result in the above theorem, we are able to deduce the following mass formula:

¹To assure only the existence of the integral canonical model for the Shimura curve, the assumption on p could be relaxed. By [5], the assumption that D is split over \mathfrak{p} is sufficient for the existence.

Theorem 1.4. *Assume $p \geq 5$ in addition to Assumption 1.1. Let $\mathcal{S} \subset M_0(\bar{k})$ be the supersingular locus. One has the equality*

$$|\mathcal{S}| = (p^r - 1)(g(M_0) - 1),$$

where $g(M_0)$ is the genus² of the Shimura curve.

In fact we have proved a bit more, namely the following cycle formula

$$\mathcal{S} = \frac{1 - p^r}{2} c_1(M_0)$$

holds up to a two torsion element in the Jacobian of $M_0 \otimes \bar{k}$ (see Corollary 5.18). The formula is expected to be true without two torsion. It is also possible to deduce the above mass formula from the congruence relation, as indicated by the work of Ihara (see §1 [16]). By the underlying work it should not be difficult to show that the special points (see Definition 1.4.1 loc. cit.) are exactly the supersingular points. If this was properly done, one deduces then that the supersingular points are actually \mathbb{F}_{q^2} -rational points of M_0 , where $\mathbb{F}_q = \mathbb{F}_{p^r}$ is the residue field of F_p . In the case of PEL type, this issue is fairly well understood (see [24]).

We now explain the strategy of the proofs of the above results in certain detail. Let \mathbb{H} be the dual of the Tate module of the generic fiber of the universal abelian scheme f , which is considered as an étale \mathbb{Z}_p -local system over M^0 . First of all we show a natural direct-tensor decomposition of $\mathbb{H} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^d}$ into *rank two* étale \mathbb{Z}_{p^d} -local systems. It specializes into a direct-tensor decomposition of the p -adic Galois representation attached to the dual of the Tate module over a closed point of M^0 . Thanks to a recent result (see Theorem 3.10) in p -adic Hodge theory, that should be due to Di Matteo ([20], see also [36]), each rank two factor is shown to be a potentially crystalline \mathbb{Q}_{p^d} -representation in the sense of Fontaine. The basic two dimensional potentially crystalline \mathbb{Q}_{p^r} -representation (which is not unramified) is denoted by V_1 , which in a sense controls the variations of the Newton polygons of the total Tate module³. We derive the classification of the possible Newton polygons in $M_0(\bar{k})$ from the admissibility condition on the associated filtered ϕ -module with a crystalline representation. After examining the Galois lattice structure, the V_1 -factor is shown to be related to a Drinfel'd \mathcal{O}_p -divisible module by the fundamental work of Breuil ([3]) (see also Kisin [17]). This establishes the tensor decomposition of the universal filtered Dieudonné module over a formal neighborhood at one point (which may not appear as the origin). To extend it, one applies the theory of deformation of p -divisible groups with Tate cycles due to Faltings (§7 [13], see also §4 [22], §1.5 [18]). On the one hand, the description of a versal deformation of a p -divisible group with prescribed Tate cycles in loc. cit. reduces the problem to a group theoretical one. On the other

²The genus is defined to be one plus the half of the summation of the degree of the canonical class of each component in $M_0 \otimes \bar{k}$.

³We expect V_1 is indeed crystalline, not only potentially crystalline. Moreover the tensor factor corresponding to it should be a dual crystalline sheaf in the sense of Faltings [12] and hence plays the role as a *uniformizing* étale \mathbb{Z}_{p^r} -local system for the Shimura curve.

hand, by the very construction of the integral canonical model (see §2 [18]) the formal neighborhood of the Shimura curve at a characteristic p closed point is naturally a versal deformation of the associated p -divisible group with Tate cycles. So we are done. In our approach to a mass formula for the supersingular locus, it is crucial to construct a variant of the Hasse-Witt map which is defined over a *line bundle* on M_0 . We have constructed a *weak Hasse-Witt map* $(\mathcal{P}_0, \tilde{F}_{rel}^r)$ over M_0 to the effect that the supersingular locus coincides with the support of the zero divisor of the section of $\mathcal{P}_0 \otimes F_{M_0}^{*r} \mathcal{P}_0^{-1}$ induced by the morphism \tilde{F}_{rel}^r . To determine the multiplicity of the degeneracy of \tilde{F}_{rel}^r at a supersingular point, we apply the theory of display ([30],[31],[41]-[42]) to a versal deformation of a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$ -divisible module. We show that the multiplicity is everywhere two, and hence the result on the mass formula.

We conclude the introduction with a speculation on a general Shimura curve of Hodge type. It concerns with the existence of a Hasse-Witt pair and its connection to Conjecture 1.3 [34]. Let $G_{\mathbb{Q}} \hookrightarrow \mathrm{GSp}_{\mathbb{Q}}$ and $(G_{\mathbb{Q}}, X)$ be a Shimura curve of Hodge type. Let $K = K_p K^p \subset G(\mathbb{Q}_p)G(\mathbb{A}_f^p)$ be a compact open subgroup and $M_K := Sh_K(G, X)$ the Shimura curve which is defined over its reflex field $E := E(G, X)$. Let p be a rational odd prime such that K_p is hyperspecial (this implies that E is unramified over p). Let \mathfrak{p} be a prime of E over p . By Kisin [18], M_K has the integral canonical model over $\mathcal{O}_{E_{\mathfrak{p}}}$, and any automorphic vector bundle over M_K has a natural integral model. His result implies also the existence a universal abelian scheme over M , the completion of the integral canonical model at \mathfrak{p} , once K^p is assumed to be small enough. Let \mathcal{L} be the integral model of the automorphic line bundle over M , that is determined by the condition that its pull-back to the universal cover of a connected component comes from the restriction of $\mathcal{O}_{\mathbb{P}^1}(1)$ on \mathbb{P}^1 , the compact dual of the unit disk (see Ch. III [25]). Let $r = [E_{\mathfrak{p}} : \mathbb{Q}_p]$ be the local degree.

Conjecture 1.5. There exists a nonzero morphism $F_{rel}^r : F_{M_0}^{*r} \mathcal{L}_0^{-1} \rightarrow \mathcal{L}_0^{-1}$ whose zero locus is reduced and coincides with *the* nongeneric Newton strata.

The pair $(\mathcal{L}_0^{-1}, F_{rel}^r)$ forms then the conjectural Hasse-Witt pair. The mass formula in Conjecture 1.3 [34] is a direct consequence of the existence. More importantly it implies that the Newton stratification does not depend on the choice of a symplectic embedding of $G_{\mathbb{Q}}$. In the case studied in the paper, we have constructed the square of this conjectural pair, which is the weak Hasse-Witt pair explained before.

Acknowledgements: This work benefits from discussions with Jean-Marc Fontaine during his stay in Mainz in April 2010. We would like to express sincere thanks to him. We would like also to thank Yi Ouyang and Liang Xiao for their helps on Theorem 3.10.

2. A SHIMURA CURVE OF HODGE TYPE

Let F be a totally real field of degree d and D a quaternion division algebra over F , which is split only at one real place of F . We denote the set of real embeddings of F by

$$\Psi = \{\tau = \tau_1, \dots, \tau_d\},$$

and assume that D is split over τ . Let $\bar{\mathbb{Q}}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} . Recall (see §4 [27]) that the corestriction $\text{Cor}_{F|\mathbb{Q}} D$ is defined as the subalgebra of $\text{Gal}_{\mathbb{Q}}$ -invariant elements of

$$\bigotimes_{i=1}^d D \otimes_{F, \tau_i} \bar{\mathbb{Q}}.$$

For it one has the following result:

Lemma 2.1 (Lemma 5.7 (a) [38]). *Let F and D be as above. It holds that either*

- (i) $\text{Cor}_{F|\mathbb{Q}}(D) \cong M_{2d}(\mathbb{Q})$ and d is an odd number ≥ 3 , or
- (ii) $\text{Cor}_{F|\mathbb{Q}}(D) \not\cong M_{2d}(\mathbb{Q})$. Then

$$\text{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}) \cong M_{2d}(\mathbb{Q}(\sqrt{b})),$$

where $\mathbb{Q}(\sqrt{b})$ is a(n) real (resp. imaginary) quadratic field extension of \mathbb{Q} if d is odd (resp. even).

Both cases can be written uniformly into

$$\text{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}) \cong M_{2d}(\mathbb{Q}(\sqrt{b}))$$

for a square free rational number $b \in \mathbb{Q}$. Such an isomorphism will be fixed. One defines a function $\epsilon(D) = 0$ when $b = 1$ (i.e. Case (i)) and $\epsilon(D) = 1$ otherwise. So we can fix an embedding of \mathbb{Q} -algebras:

$$\text{Cor}_{F|\mathbb{Q}}(D) \hookrightarrow M_{2d+\epsilon(D)}(\mathbb{Q}).$$

Recall also that one comes along with a natural morphism of \mathbb{Q} -groups:

$$\text{Nm} : D^* \rightarrow \text{Cor}_{F|\mathbb{Q}}(D)^*, \quad d \mapsto (d \otimes 1) \otimes \dots \otimes (d \otimes 1).$$

So one obtains a linear representation of the \mathbb{Q} -group D^* :

$$\text{Nm} : D^* \rightarrow \text{GL}_{2d+\epsilon(D), \mathbb{Q}}.$$

It gives rise to a Shimura curve of Hodge type. We state its construction without proof and leave the details to the reader. Put $\tilde{G}'_{\mathbb{Q}} := \{x \in D \mid \text{Norm}(x) = 1\}$ and $\tilde{G}_{\mathbb{Q}} := \mathbb{G}_{m, \mathbb{Q}} \times \tilde{G}'_{\mathbb{Q}}$, and write $\text{GL}_{\mathbb{Q}}$ for $\text{GL}_{2d+\epsilon(D), \mathbb{Q}}$. The \mathbb{Q} -group $G_{\mathbb{Q}}$ is defined to be image of the morphism $\tilde{G}_{\mathbb{Q}} \rightarrow \text{GL}_{\mathbb{Q}}$, that is the product of the natural morphism $\mathbb{G}_{m, \mathbb{Q}} \rightarrow \text{GL}_{\mathbb{Q}}$ and $\text{Nm}|_{\tilde{G}'_{\mathbb{Q}}}$. It is connected and reductive. The natural morphism $N : \tilde{G}_{\mathbb{Q}} \rightarrow G_{\mathbb{Q}}$ is a central isogeny. Let $G'_{\mathbb{Q}}$ be the image of $\tilde{G}'_{\mathbb{Q}}$ in $G_{\mathbb{Q}}$. The natural embedding $G_{\mathbb{Q}} \hookrightarrow \text{GL}_{\mathbb{Q}}$ factors through $\text{GSp}_{\mathbb{Q}} \subset \text{GL}_{\mathbb{Q}}$, which can be seen as follows: Let $H_{\mathbb{Q}} := \mathbb{Q}(\sqrt{b})^{2d}$ be a \mathbb{Q} -vector space with the $\text{Cor}_{F|\mathbb{Q}}(D)$ action by the left multiplication and $\mathbb{G}_{m, \mathbb{Q}}$ action by scalar multiplication. This induces a $G_{\mathbb{Q}}$ -action on $H_{\mathbb{Q}}$. It is easy to verify that there exists a $\mathbb{Q}(\sqrt{b})$ -valued

symplectic form ω on $H_{\mathbb{Q}}$, unique up to scalar, which is invariant under the $G'_{\mathbb{Q}}$ -action. Then $G_{\mathbb{Q}} = \mathbb{G}_{m,\mathbb{Q}} \cdot \tilde{G}'_{\mathbb{Q}} \subset \mathrm{GL}_{\mathbb{Q}}$ acts on the \mathbb{Q} -valued symplectic form

$$\psi := \mathrm{tr}_{\mathbb{Q}(\sqrt{b})|\mathbb{Q}} \omega$$

by similitude. Let S^1 be the real group $\{z \in \mathbb{C} \mid z\bar{z} = 1\}$. One defines

$$u_0 : S^1 \rightarrow \tilde{G}'_{\mathbb{R}}(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SU}(2)^{\times d-1}, \quad e^{i\theta} \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \times id^{\times d-1}.$$

The morphism $\tilde{h}_0 = id \times u_0 : \mathbb{R}^* \times S^1 \rightarrow \tilde{G}'_{\mathbb{R}}$ descends to a morphism of real groups:

$$h_0 : \mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m \rightarrow G_{\mathbb{R}}.$$

Let X be the $G(\mathbb{R})$ -conjugacy class of h_0 and $(\mathrm{GSp}(H_{\mathbb{Q}}, \psi), X(\psi))$ the Siegel space defined by $(H_{\mathbb{Q}}, \psi)$. One verifies that

$$(G_{\mathbb{Q}}, X) \hookrightarrow (\mathrm{GSp}(H_{\mathbb{Q}}, \psi), X(\psi))$$

is a morphism of Shimura datum, and therefore defines a Shimura curve of Hodge type. Note that only for $d = 1, 2$ it is a Shimura curve of PEL type (see Lemma 5.9 in [38]). The case $d = 3$ and $\epsilon(D) = 0$ is the original example considered by D. Mumford in loc. cit.. Now let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup and one defines the Shimura curve as the double coset

$$Sh_K(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K,$$

where

$$q(x, a)b = (qx, qab), \quad q \in G(\mathbb{Q}), x \in X, a \in G(\mathbb{A}_f), b \in K.$$

By the theory of canonical model (see e.g. [25]), $M_K := Sh_K(G, X)$ is naturally defined over the reflex field of (G, X) , that is $\tau(F) \subset \mathbb{C}$ in this current case (see Example 12.4 (d) [26]). It is not difficult to show that M_K is proper over F .

Now let p be a rational prime satisfying the following assumption:

Assumption 2.2. p is an odd prime and does not divide the discriminants of F and D .

Let $\mathcal{O} := \mathcal{O}_F$ be the ring of algebraic integers, and

$$p\mathcal{O} = \prod_{i=1}^n \mathfrak{p}_i,$$

the prime decomposition. By choosing an embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ (which is then fixed once for all), one gets an identification of Ψ with

$$\mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}}_p) = \prod_{i=1}^n \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \bar{\mathbb{Q}}_p).$$

Write F_i for $F_{\mathfrak{p}_i}$ and put $r_i := [F_i : \mathbb{Q}_p]$. We can assume that $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_1}, \bar{\mathbb{Q}}_p)$. Set $\mathfrak{p} = \mathfrak{p}_1$ and $r = r_1$. The condition on p implies that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension of \mathbb{Q}_p . Hence hyperspecial subgroups exist in $G(\mathbb{Q}_p)$ (see 1.10 [35]). Recall that we have a central isogeny $\tilde{G}_{\mathbb{Q}} \rightarrow G_{\mathbb{Q}} \subset \mathrm{GL}_{\mathbb{Q}}$ over \mathbb{Q}

with $\tilde{G}_{\mathbb{Q}} = \mathbb{G}_{m,\mathbb{Q}} \times \tilde{G}'_{\mathbb{Q}}$, where $\tilde{G}'_{\mathbb{Q}} = \ker(\text{Norm} : D^* \rightarrow F^*)$. The assumption on p implies that

$$\begin{aligned} D^*(\mathbb{Q}_p) &= (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^* \\ &= \prod_{i=1}^n (D \otimes_F F_{\mathfrak{p}_i})^* \\ &\cong \prod_{i=1}^n \text{GL}_2(F_{\mathfrak{p}_i}). \end{aligned}$$

It should be clear that $\text{Norm} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ becomes a product of the determinants under the above identification. So this implies that $\tilde{G}'(\mathbb{Q}_p) \cong \prod_{i=1}^n \text{SL}_2(F_{\mathfrak{p}_i})$, and hence

$$\tilde{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^* \times \prod_{i=1}^n \text{SL}_2(F_{\mathfrak{p}_i}).$$

Thus a hyperspecial subgroup of $G(\mathbb{Q}_p)$ is conjugate to the image of

$$\mathbb{Z}_p^* \times \prod_{i=1}^n \text{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}}) \subset \tilde{G}(\mathbb{Q}_p)$$

under the isogeny $\tilde{G}(\mathbb{Q}_p) \rightarrow G(\mathbb{Q}_p)$. In what follows the p -component $K_p \subset G(\mathbb{Q}_p)$ of the level structure $K (= K_p K^p \subset G(\mathbb{Q}_p)G(\mathbb{A}_f^p))$ is always taken to be hyperspecial. The main result of Kisin [18] asserts that, for our chosen prime $\mathfrak{p}|p$, there exists *the* integral canonical model \mathcal{M}_K of M_K , which is a smooth $\mathcal{O}_{(\mathfrak{p})}$ scheme for K^p sufficiently small. The construction of \mathcal{M}_K (see §2.3 loc. cit.) provides with a universal abelian scheme over \mathcal{M}_K as well, once the coprime to p -component K^p is chosen small enough: Take a suitable maximal order \mathcal{O}_D of the F -algebra D and consider

$$\text{Cor}_{F|\mathbb{Q}} \mathcal{O}_D := (\otimes_{i=1}^d \mathcal{O}_D \otimes_{\mathcal{O}_F, \tau_i} \bar{\mathbb{Z}})^{\text{Gal}_{\mathbb{Q}}} \subset \text{Cor}_{F|\mathbb{Q}} D.$$

This gives rise to a lattice $H_{\mathbb{Z}} = \text{Cor}_{F|\mathbb{Q}} \mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Q}(\sqrt{b})} \subset H_{\mathbb{Q}}$ such that there is a closed embedding $G_{\mathbb{Z}_p} \hookrightarrow \text{GL}(H_{\mathbb{Z}_p})$ (where $G_{\mathbb{Z}_p}$ is the reductive group scheme over \mathbb{Z}_p associated with K_p) whose generic fiber is the base change to \mathbb{Q}_p of $G_{\mathbb{Q}} \hookrightarrow \text{GL}_{\mathbb{Q}}$. Let $K'_p \subset \text{GSp}(\mathbb{Q}_p)$ be the stabilizer of $H_{\mathbb{Z}_p}$. One can choose a $K'^p \subset \text{GSp}(\mathbb{A}_f^p)$ such that for $K' = K'_p K'^p$, one has an embedding of Shimura varieties

$$M_K \hookrightarrow Sh_{K'}(\text{GSp}(\psi), X(\psi)),$$

(which is defined over F), and $Sh_{K'}(\text{GSp}(\psi), X(\psi))$ has an integral model

$$\mathcal{S}_{K'} := \mathcal{S}_{K'}(\text{GSp}(\psi), X(\psi))$$

over $\mathbb{Z}_{(p)}$ (which is not necessarily smooth) representing a moduli functor over $\mathbb{Z}_{(p)}$ (see §2.3.3 [18]). As K^p is required to be small enough, one may further assume that K'^p is taken so small that there exists an abelian scheme $\mathcal{A}_{K'} \rightarrow \mathcal{S}_{K'}$ over $\mathcal{S}_{K'}$. Recall (see Theorem 2.3.8 loc. cit.) that \mathcal{M}_K is defined as the normalization of the closure of the composite

$$M_K \hookrightarrow Sh_{K'}(\text{GSp}(\psi), X(\psi)) \hookrightarrow \mathcal{S}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\mathfrak{p})}.$$

Now we define our abelian scheme $f_K : \mathcal{X}_K \rightarrow \mathcal{M}_K$ to be the morphism sitting in the Cartesian diagram:

$$\begin{array}{ccc} \mathcal{X}_K & \longrightarrow & \mathcal{A}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\mathfrak{p})} \\ f_K \downarrow & & \downarrow \\ \mathcal{M}_K & \longrightarrow & \mathcal{S}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\mathfrak{p})}. \end{array}$$

For the sake of convenience, we shall change our foregoing notation as follows: Let M (resp. $f : X \rightarrow M$) be the completion of the integral canonical model \mathcal{M}_K (resp. $f_K : \mathcal{X}_K \rightarrow \mathcal{M}_K$) at \mathfrak{p} . They are defined over the discrete valuation ring $\mathcal{O}_{\mathfrak{p}}$, the completion of $\mathcal{O}_{(\mathfrak{p})}$ at the maximal ideal. The superscript (resp. subscript) zero on an object means the base change of the object to the generic (resp. closed) fiber of $\mathcal{O}_{\mathfrak{p}}$. For example $f_0 : X_0 \rightarrow M_0$ is the reduction of f modulo \mathfrak{p} . For a k -rational point x_0 of M_0 , the closed fiber of f_0 at x_0 is denoted by A_{x_0} . The Newton polygon of A_{x_0} is defined to be the Newton polygon of its associated p -divisible group. The first part of this paper is to study the possible Newton polygons which the A_{x_0} may take when $x_0 \in M_0(\bar{k})$ varies. A Newton strata is defined to be the sublocus of $M_0(\bar{k})$ with the same Newton polygon. By the theorem of Grothendieck-Katz, the Newton polygon of A_{x_0} jumps under specialization. So a nongeneric Newton strata forms a proper closed subset, which in the current case means a reduced divisor of $M_0 \otimes \bar{k}$. By the result of the first part, we know that there is *only* one nongeneric Newton strata. The second part is to count the number of supersingular points, which is a global question in nature.

3. TWO DIMENSIONAL POTENTIALLY CRYSTALLINE \mathbb{Q}_{p^r} -REPRESENTATIONS AND CLASSIFICATION OF THE NEWTON POLYGONS

Let $f : X \rightarrow M$ be the universal abelian scheme as above, which is defined over $\mathcal{O}_{\mathfrak{p}}$. First of all, we introduce two sides of the whole matter: Let

$$\mathbb{H} := R^1 f_*^0(\mathbb{Z}_p)_{\bar{X}_{et}^0}$$

be the étale \mathbb{Z}_p -local system over M^0 and

$$(H, F, \nabla, \phi) := (H_{dR}^1(X|M), F_{hod}, \nabla^{GM}, \phi_{rel})$$

the universal filtered Dieudonné module. The latter is an object of the Faltings's category $MF_{[0,1]}^{\nabla}(M)$ defined in §4 [13]. We denote by $MF_{[0,a]}^{\nabla}(M)_{tor}$ the p torsion analogue of the previous category that was introduced in §2 [12]. In loc. cit. Faltings has constructed a fully faithful functor \mathbf{D} from $MF_{[0,p-2]}^{\nabla}(M)$ (resp. $MF_{[0,p-2]}^{\nabla}(M)_{tor}$) to the category of étale \mathbb{Z}_p (resp. p -torsion) local systems over M^0 . By the Remark after Theorem 2.6* in [12], one has $\mathbf{D}(\mathcal{O}_X/p^n, d) = \mathbb{Z}/p^n$ for each $n \in \mathbb{N}$. Applying Theorem 6.2 in loc. cit. about the compatibility of the direct image with the functor \mathbf{D} to f and $(\mathcal{O}_X/p^n, d)$, one sees that

$$\mathbf{D}(H/p^n, F, \nabla, \phi) = \mathbb{H}^*/p^n.$$

Taking the inverse limit, one obtains that

$$\mathbf{D}(H, F, \nabla, \phi) = \mathbb{H}^*.$$

3.1. Tensor decomposition of the Galois representation. Let x_0 be a k -rational point of M_0 . Because M is smooth over $\mathcal{O}_{\mathfrak{p}}$, there exists an $\mathcal{O}_{W(k)}$ -valued point x of M which lifts x_0 . Let A_x be the corresponding abelian scheme over $\mathcal{O}_{W(k)}$ whose reduction is equal to A_{x_0} . The aim of this paragraph is to show a certain tensor decomposition of the p -adic Galois representation associated with the generic fiber A_{x^0} of A_x .

Lemma 3.1. *One has a natural isomorphism of \mathbb{Q}_p -algebras:*

$$\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \otimes_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i).$$

Proof. Put $D_i = D \otimes_{F, \tau_i} \bar{\mathbb{Q}}, 1 \leq i \leq d$. For an element $a \otimes \lambda \in D_i$ and $g \in \mathrm{Gal}_{\mathbb{Q}}$,

$$g(a \otimes_{\tau_i} \lambda) = a \otimes_{g(\tau_i)} g(\lambda).$$

By the definition of the corestriction, one has a natural isomorphism of $\mathrm{Gal}_{\mathbb{Q}}$ -modules:

$$\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \cong \otimes_{i=1}^d D_i.$$

Let D_{ι} be the decomposition group of ι in $\mathrm{Gal}_{\mathbb{Q}}$, which is isomorphic to the local Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$. Compare the D_{ι} -invariants of two sides of the above isomorphism after tensoring with $\bar{\mathbb{Q}}_p$ via ι : One obtains the $\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ from the left side. Let

$$O_1 := \{\tau_1 = \tau, \dots, \tau_{r_1} = \tau_r\}, \dots, O_n = \{\tau_{r_1+\dots+r_{d-1}+1}, \dots, \tau_{r_1+\dots+r_{d-1}+r_d} = \tau_n\}$$

be the n -orbits of D_{ι} -action on Ψ . Note that there is a natural isomorphism

$$(\otimes_{\tau_j \in O_i} D_j \otimes_{\bar{\mathbb{Q}}, \iota} \bar{\mathbb{Q}}_p)^{D_{\iota}} \cong \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i).$$

As the tensor product $\otimes_{i=1}^n (\otimes_{\tau_j \in O_i} D_j \otimes_{\bar{\mathbb{Q}}, \iota} \bar{\mathbb{Q}}_p)^{D_{\iota}}$ over \mathbb{Q}_p is clearly a subspace of the D_{ι} -invariants on the right side, it is the whole invariant space for the dimension reason. Now the lemma follows. \square

Consider the base change to \mathbb{Q}_p of the \mathbb{Q} -morphism $\mathrm{Nm} : D^* \rightarrow \mathrm{Cor}_{F|\mathbb{Q}}(D)^*$. The following statement is clear from the proof of the last lemma.

Lemma 3.2. *The morphism $\mathrm{Nm}_{\mathbb{Q}_p} : D^*(\mathbb{Q}_p) \rightarrow \mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)$ factors through the natural morphism*

$$\prod_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^* \rightarrow \mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p).$$

Moreover, under the natural decomposition $D^*(\mathbb{Q}_p) = \prod_{i=1}^n (D \otimes_F F_i)^*$, $\mathrm{Nm}_{\mathbb{Q}_p}$ is written into a product $\prod_{i=1}^n \mathrm{Nm}_i$ such that for each i , the morphism

$$\mathrm{Nm}_i : (D \otimes_F F_i)^* \rightarrow \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^*$$

is the natural diagonal morphism for the corestriction.

As a consequence, the representation of $D^*(\mathbb{Q}_p)$ on $H_{\mathbb{Q}_p}$ admits a natural tensor decomposition: By Schur's lemma the representation decomposes as a tensor product. In the concrete situation, this can be seen in a direct way: By assumption 2.2, $D \otimes_F F_i$ splits for each i , and so does $\text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)$ (which is isomorphic to $M_{2r_i}(\mathbb{Q}_p)$). In the case of $\epsilon(D) = 0$, the morphism

$$\prod_{i=1}^n \text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^* \rightarrow \text{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p) = \text{GL}(H_{\mathbb{Q}_p})$$

is isomorphic to the tensor product morphism

$$\prod_{i=1}^n \text{GL}_{2r_i}(\mathbb{Q}_p) \longrightarrow \text{GL}_{2d}(\mathbb{Q}_p), (g_1, \dots, g_n) \mapsto g_1 \otimes \dots \otimes g_n.$$

If $\epsilon(D) = 1$, then $\text{Cor}_{F|\mathbb{Q}}(D)$ is nonsplit and it splits after tensoring with $\mathbb{Q}(\sqrt{b})$. Consider the composite

$$\begin{aligned} \prod_{i=1}^n \text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^* &\rightarrow \text{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p) \subset (\text{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}))^*(\mathbb{Q}_p) \\ &= \text{GL}_{\mathbb{Q}(\sqrt{b})}(H_{\mathbb{Q}})(\mathbb{Q}_p) \subset \text{GL}(H_{\mathbb{Q}})(\mathbb{Q}_p). \end{aligned}$$

The above inclusion $\text{Cor}_{F|\mathbb{Q}}(D)^* \subset (\text{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}))^*$ is given by $a \mapsto a \otimes 1$. One has the following commutative diagram:

$$\begin{array}{ccc} \text{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p) \times \mathbb{Q}(\sqrt{b})^*(\mathbb{Q}_p) & \longrightarrow & \text{GL}_{\mathbb{Q}(\sqrt{b})}(H_{\mathbb{Q}})(\mathbb{Q}_p) \\ \cap \downarrow & & \downarrow \cap \\ \text{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p) \times \text{GL}_2(\mathbb{Q}_p) & \longrightarrow & \text{GL}(H_{\mathbb{Q}})(\mathbb{Q}_p) \end{array}$$

The image of $\prod_{i=1}^n \text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^*$ in the left-up element of the above diagram is contained in $\text{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p) \times \{1\}$. Thus the morphism

$$\prod_{i=1}^n \text{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i)^* \rightarrow \text{GL}(H_{\mathbb{Q}_p})$$

is isomorphic to the composite of the obvious morphisms

$$\prod_{i=1}^n \text{GL}_{2r_i}(\mathbb{Q}_p) \hookrightarrow \prod_{i=1}^n \text{GL}_{2r_i}(\mathbb{Q}_p) \times \text{GL}_2(\mathbb{Q}_p) \xrightarrow{\otimes} \text{GL}_{2d+1}(\mathbb{Q}_p).$$

For $s \in \mathbb{N}$, one denotes by $\sigma \in \text{Gal}(\mathbb{Q}_{p^s}|\mathbb{Q}_p)$ the Frobenius automorphism. For a topological group P with a continuous \mathbb{Z}_{p^s} linear representation W , the σ^i -conjugate W_{σ^i} of W for $0 \leq i \leq s-1$, is defined to be the tensor product

$$W \otimes_{\mathbb{Z}_{p^s}, \sigma^i} \mathbb{Z}_{p^s},$$

where P acts on \mathbb{Z}_{p^s} trivially. Similarly define the σ^i -conjugations of a \mathbb{Q}_{p^s} -representation. The symbol \otimes_{σ^i} signifies the equalities of two tensors:

$$\lambda(x \otimes \mu) = x \otimes \lambda\mu, \quad \lambda x \otimes \mu = x \otimes \lambda^{\sigma^i} \mu, \quad \text{for } \lambda, \mu \in \mathbb{Z}_{p^s}, x \in W.$$

Consider the morphism

$$\mathrm{Nm}_1 : (D \otimes_F F_1)^* \longrightarrow \mathrm{Cor}_{F_1|\mathbb{Q}_p}(D \otimes_F F_1)^*, \quad a \mapsto (a \otimes_{F_1, \tau_1} 1) \otimes \cdots \otimes (a \otimes_{F_1, \tau_r} 1).$$

Note that for $1 \leq i \leq r$, $\tau_i(F_1) = \mathbb{Q}_{p^r}$, the unique unramified extension of \mathbb{Q}_p of degree r in $\bar{\mathbb{Q}}_p$. Then one has a natural isomorphism

$$\mathrm{Cor}_{F_1|\mathbb{Q}_p}(D \otimes_F F_1) \otimes_{\mathbb{Q}_p} F_1 \cong \otimes_{i=0}^{r-1} (D \otimes_F F_1 \otimes_{F_1, \sigma^i} F_1).$$

This implies that the natural morphism

$$(D \otimes_F F_1)^* \longrightarrow \mathrm{Cor}_{F_1|\mathbb{Q}_p}(D \otimes_F F_1)^*(F_1)$$

is isomorphic to the composite of

$$\mathrm{GL}_2(\mathbb{Q}_{p^r}) \hookrightarrow \prod_{i=0}^{r-1} \mathrm{GL}_2(\mathbb{Q}_{p^r}), \quad g \mapsto (g, \dots, \sigma^{r-1}(g))$$

with the tensor product morphism

$$\prod_{i=0}^{r-1} \mathrm{GL}_2(\mathbb{Q}_{p^r}) \longrightarrow \mathrm{GL}_{2r}(\mathbb{Q}_{p^r}).$$

Summarizing the above discussions, we derive the following

Lemma 3.3. *The representation of $D^*(\mathbb{Q}_p)$ on $H_{\mathbb{Q}_p}$ admits a natural tensor decomposition:*

$$H_{\mathbb{Q}_p} = (V_{\mathbb{Q}_p} \otimes U_{1, \mathbb{Q}_p} \otimes \cdots \otimes U_{n-1, \mathbb{Q}_p})^{\oplus 2^{\epsilon(D)}}.$$

Moreover, the representation $V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$ decomposes further into a tensor product

$$V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1, \sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1, \sigma^{r-1}}$$

with $\dim_{\mathbb{Q}_{p^r}} V_1 = 2$. For $1 \leq i \leq n-1$, $U_{i, \mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^{r_{i+1}}}$ decomposes into a tensor product in a similar way.

Lemma 3.4. *Let P be a topological group together with a continuous linear representation on a finite dimensional \mathbb{Q}_{p^s} -vector space W . Assume the following two conditions hold:*

(i) *The representation factors as*

$$P \rightarrow \mathrm{GL}(W_1) \times \mathrm{SL}(W_2) \xrightarrow{\otimes} \mathrm{GL}(W),$$

where $W_i, i = 1, 2$ are two \mathbb{Q}_{p^s} -vector spaces.

(ii) *There is a \mathbb{Z}_{p^s} -lattice $W_{\mathbb{Z}_{p^s}}$ in W , which is stable under the P -action and admits a lattice tensor decomposition*

$$W_{\mathbb{Z}_{p^s}} = W_{1, \mathbb{Z}_{p^s}} \otimes_{\mathbb{Z}_{p^s}} W_{2, \mathbb{Z}_{p^s}},$$

where $W_{i, \mathbb{Z}_{p^s}}$ is a \mathbb{Z}_{p^s} -lattice of W_i for $i = 1, 2$.

Then in the factorization of (i), the lattice $W_{i, \mathbb{Z}_{p^s}}$ for $i = 1, 2$ is stable under the P -action on W_i .

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
& & \mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}}) & \xrightarrow{\otimes} & \mathrm{GL}(W_{\mathbb{Z}_{p^s}}) \\
& & \downarrow \cap & & \downarrow \cap \\
P & \longrightarrow & \mathrm{GL}(W_1) \times \mathrm{SL}(W_2) & \xrightarrow{\otimes} & \mathrm{GL}(W)
\end{array}$$

It suffices to show that the representation $P \rightarrow \mathrm{GL}(W_{\mathbb{Z}_{p^s}}) \subset \mathrm{GL}(W)$ factors through

$$\mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}}) \rightarrow \mathrm{GL}(W_{\mathbb{Z}_{p^s}}).$$

Note that $\mathrm{GL}(W_{\mathbb{Z}_{p^s}})$ is a compact subgroup of $\mathrm{GL}(W)$. As the morphism \otimes has a finite kernel,

$$T := \otimes^{-1}(\mathrm{GL}(W_{\mathbb{Z}_{p^s}}) \cap \otimes(\mathrm{GL}(W_1) \times \mathrm{SL}(W_2)))$$

is a compact subgroup of $\mathrm{GL}(W_1) \times \mathrm{SL}(W_2)$. Since T contains $\mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}})$, which is maximal compact, it holds that $T = \mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}})$. Since by assumptions the image of P in $\mathrm{GL}(W_1) \times \mathrm{SL}(W_2)$ is contained in T , the morphism $P \rightarrow \mathrm{GL}(W_{\mathbb{Z}_{p^s}})$ factors through

$$\mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}}) \rightarrow \mathrm{GL}(W_{\mathbb{Z}_{p^s}}).$$

This proves the lemma. \square

Proposition 3.5. *One has a natural tensor decomposition of the K_p -representation on $H_{\mathbb{Z}_p}$:*

$$H_{\mathbb{Z}_p} = (V \otimes U)^{\oplus 2^{\epsilon(D)}}, U = U_1 \otimes \cdots \otimes U_{n-1}$$

and furthermore

$$V \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = V_1 \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma^{r-1}},$$

and similarly for each U_i .

Proof. Recall that K_p is conjugate to the image of

$$\tilde{K}_p := (\mathbb{Z}_p^* \times \mathrm{SL}_2(\mathcal{O}_{F_p})) \times \prod_{i=2}^n \mathrm{SL}_2(\mathcal{O}_{F_{p_i}}) \subset \tilde{G}(\mathbb{Q}_p)$$

under the map $N_{\mathbb{Q}_p} : \tilde{G}(\mathbb{Q}_p) \rightarrow G(\mathbb{Q}_p)$. The tensor decomposition of $H_{\mathbb{Q}_p}$ as $D^*(\mathbb{Q}_p)$ -representation in Lemma 3.3 induces a tensor decomposition of $H_{\mathbb{Q}_p}$ as \tilde{K}_p -representation. Since the \tilde{K}_p -action on $H_{\mathbb{Q}_p}$ factors through the K_p -action on $H_{\mathbb{Q}_p}$ by definition, one obtains the tensor decomposition of $H_{\mathbb{Q}_p}$ for the K_p -action as well. By the definition of the lattice $H_{\mathbb{Z}}$, it is easy to see that $H_{\mathbb{Z}_p}$ decomposes into a tensor product of \mathbb{Z}_p -lattices as in the above statement. Then it is also a tensor decomposition as K_p -representation by Lemma 3.4. The proofs of the tensor decompositions for the V and U factors are analogous. \square

By Proposition 2.2.4 [18], each geometrically connected component of M^0 is defined over an unramified extension of F_p . Since there is a finite number of them, we fix an $L \subset \mathbb{Q}_p^{ur}$, which is a finite extension of F_p and over which each component defines. Also we can further assume that each component has an L -rational point.

Corollary 3.6. *One has a natural tensor decomposition of étale local systems over $M^0 \times_{F_p} L$:*

$$\mathbb{H} = (\mathbb{V} \otimes \mathbb{U})^{\oplus 2^{\epsilon(D)}}, \mathbb{U} = \mathbb{U}_1 \otimes \cdots \otimes \mathbb{U}_{n-1}$$

and

$$\mathbb{V} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} \cong \mathbb{V}_1 \otimes_{\mathbb{Z}_{p^r}} \mathbb{V}_{1,\sigma} \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} \mathbb{V}_{1,\sigma^{r-1}},$$

where for $0 \leq i \leq r-1$, \mathbb{V}_{1,σ^i} is the σ^i -conjugate of \mathbb{V}_1 . Similarly for \mathbb{U}_i , one has

$$\mathbb{U}_i \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^{r_i}} \cong \mathbb{U}_{i,1} \otimes_{\mathbb{Z}_{p^{r_i}}} \mathbb{U}_{i,\sigma} \otimes_{\mathbb{Z}_{p^{r_i}}} \cdots \otimes_{\mathbb{Z}_{p^{r_i}}} \mathbb{U}_{i,\sigma^{r_i-1}}.$$

Proof. Write $M^0 \times_{F_p} L = \bigsqcup_i M_i^0$ the disjoint union of its geometrically connected components, and let M_1^0 be the component which is represented by the double coset $[1] \in G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f) / K$. It suffices to show that the tensor decomposition of the restriction \mathbb{H} to M_1^0 . Consider the short exact sequence of étale fundamental groups:

$$1 \rightarrow \pi_1^{geo}(M_1^0) \rightarrow \pi_1^{arith}(M_1^0) \rightarrow \text{Gal}(\bar{\mathbb{Q}}_p | L) \rightarrow 1.$$

An L -rational point of M^0 induces a splitting of the exact sequence, and one writes

$$\pi_1^{arith}(M_1^0) = \pi_1^{geo}(M_1^0) \cdot \text{Gal}(\bar{\mathbb{Q}}_p | L).$$

To show the tensor decomposition of $\mathbb{H}|_{M_1^0}$, it is to show that the factorization of $\pi_1^{geo}(M_1^0) \rightarrow K_p$ and $\text{Gal}(\bar{\mathbb{Q}}_p | L) \rightarrow K_p$. The latter follows from the proof of Lemma 2.2.1 [18]. The former goes as follows: It is known that $\pi_1^{top}(M_1^0(\mathbb{C}))$ is equal to $K \cap G(\mathbb{Q})_+$. The representation $\pi_1^{geo}(M_1^0) \rightarrow \text{GL}(H_{\mathbb{Z}_p})$ is the composite of

$$\pi_1^{geo}(M_1^0) \cong \hat{\pi}_1^{top}(M_1^0) \xrightarrow{\hat{\cdot}} \text{GL}(H_{\mathbb{Z}}) \twoheadrightarrow \text{GL}(H_{\mathbb{Z}_p}).$$

Obviously the representation $\pi_1^{top}(M_1^0) = K \cap G(\mathbb{Q})_+ \rightarrow \text{GL}(H_{\mathbb{Z}})$ factors through $K \subset \text{GL}(H_{\mathbb{Z}})$. Hence the result follows from Proposition 3.5. \square

Specializing the above tensor decompositions of étale local systems into a closed point, one obtains the following

Corollary 3.7. *Let E be a finite extension of L and x^0 an E -rational point of M^0 . Let $H_{\mathbb{Z}_p} = H_{et}^1(\bar{A}_{x^0}, \mathbb{Z}_p)$ and*

$$\rho : \text{Gal}_E \rightarrow \text{GL}(H_{\mathbb{Z}_p})$$

be the associated Galois representation. Then one has a tensor decomposition as Gal_E -module:

$$H_{\mathbb{Z}_p} = (V_{\mathbb{Z}_p} \otimes U_{\mathbb{Z}_p})^{\oplus 2^{\epsilon(D)}},$$

together with a further tensor decomposition of $V_{\mathbb{Z}_p}$:

$$V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = V_1 \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma^{r-1}}.$$

3.2. Each tensor factor is potentially crystalline. It is standard that $H_{\mathbb{Q}_p}$ is a polarisable crystalline representation of Hodge-Tate weights $\{0, 1\}$. We show that each factor appearing in the tensor decomposition of Corollary 3.7 is potentially crystalline.

Proposition 3.8. *$U_{\mathbb{Q}_p}$ is a potentially unramified representation. As a consequence, both $V_{\mathbb{Q}_p}$ and $U_{\mathbb{Q}_p}$ are potentially crystalline.*

Proof. Let $I_E \subset \text{Gal}_E$ be the inertia group. We claim that the image of I_E in $\text{GL}(U_{\mathbb{Q}_p})$ is finite. Assuming the claim, one sees that $U_{\mathbb{Q}_p}$ is potentially unramified and hence potentially crystalline. Clearly $V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p}$, as a direct factor of $H_{\mathbb{Q}_p}$, is crystalline. Therefore $V_{\mathbb{Q}_p}$, that is a subrepresentation of $V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p}^*$, is also potentially crystalline. To show the claim, we introduce the Hodge-Tate cocharacter induced by the Hodge-Tate decomposition of $H_{\mathbb{Q}_p}$

$$\mu_{HT} : \mathbb{G}_m(\mathbb{C}_p) \rightarrow G(\mathbb{C}_p) \subset \text{GL}(H_{\mathbb{C}_p}),$$

and the Hodge-de Rham cocharacter

$$\mu_{HdR} : \mathbb{G}_m(\mathbb{C}) \rightarrow G(\mathbb{C}) \subset \text{GL}(H_{\mathbb{C}})$$

induced by the Hodge decomposition of $H_B^1(A(\mathbb{C}), \mathbb{Q})$. Let C_{HdR} (resp. C_{HT}) be the $G(\mathbb{C})$ (resp. $G(\mathbb{C}_p)$)-conjugacy class of μ_{HdR} (resp. μ_{HT}). Then C_{HdR} is defined over the reflex field $\tau(F) \subset \mathbb{C}$ of (G, X) . It follows from a result of Blasius and Wintenberger (see Theorem 0.3 [2] and Proposition 7 [40], see also Theorem 4.2 [32]) that

$$C_{HT} = C_{HdR} \otimes_{F, \tau} \mathbb{C}_p,$$

where $\tau : F \rightarrow \mathbb{C}_p$ is the composite

$$F \xrightarrow{\tau} \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p \subset \hat{\bar{\mathbb{Q}}}_p = \mathbb{C}_p.$$

Since $\tilde{G} \rightarrow G$ is a central isogeny, there is a natural number a such that the a -th power μ_{HdR}^a (resp. μ_{HT}^a) lifts to a cocharacter into $\tilde{G}(\mathbb{C})$ (resp. $\tilde{G}(\mathbb{C}_p)$). Consider the projection of μ_{HdR}^a to an SL_2 -factor in the decomposition

$$\tilde{G}(\mathbb{C}) = \mathbb{C}^* \times \text{SL}_2(\mathbb{C}) \times \cdots \times \text{SL}_2(\mathbb{C}),$$

where the order of SL_2 -factors is arranged according to Ψ . By the definition of \tilde{h}_0 in §2, one sees that *only* the projection to the first SL_2 -factor (corresponding to τ) is nontrivial. By the above identification, the same situation holds for the projections of μ_{HT}^a to SL_2 -factors in the decomposition

$$\tilde{G}(\mathbb{C}_p) = \mathbb{C}_p^* \times \text{SL}_2(\mathbb{C}_p) \times \cdots \times \text{SL}_2(\mathbb{C}_p),$$

where the order of SL_2 -factors is arranged according to $\text{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}}_p)$ which is identified with Ψ . By the construction of U -factor, the projection of μ_{HT}^a to the factor $\text{GL}(U_{\mathbb{C}_p})$ is trivial. So the projection of μ_{HT} to $\text{GL}(U_{\mathbb{C}_p})$ is finite. By S. Sen's theorem (see [33]), the Zariski closure of $\rho(I_E) \subset G(\mathbb{Q}_p)$ is equal to the \mathbb{Q}_p -Zariski closure of μ_{HT} . So the image of I_E in $\text{GL}(U_{\mathbb{Q}_p})$ is finite. \square

Let E be a finite field extension of \mathbb{Q}_p in general, $E_0 \subset E$ the maximal unramified subextension and r an arbitrary natural number. Recall that

Definition 3.9. A \mathbb{Q}_{p^r} -representation of Gal_E is a finite dimensional \mathbb{Q}_{p^r} -vector space V equipped with a continuous action

$$\text{Gal}_E \times V \rightarrow V$$

satisfying

$$g(v_1 + v_2) = g(v_1) + g(v_2), \quad g(\lambda v) = g(\lambda)g(v)$$

where $g \in \text{Gal}_E$, and $\lambda \in \mathbb{Q}_{p^r}$, $v, v_1, v_2 \in V$. It is called *Hodge-Tate* (resp. *de-Rham*, *crystalline*) \mathbb{Q}_{p^r} -representation if it is so as a \mathbb{Q}_p -representation.

The following result is known among experts. A variant of it was communicated by L. Berger to the first named author during the p -adic Hodge theory workshop in ICTP, 2009. The first official proof should appear in the Ph.D thesis of G. Di Matteo (see the recent preprint [20]). Another proof has been communicated to us by L. Xiao (see [36]).

Theorem 3.10. *Let V and W be two \mathbb{Q}_{p^r} -representations of Gal_E . If $V \otimes_{\mathbb{Q}_{p^r}} W$ is de Rham, and one of the tensor factors is Hodge-Tate, then each tensor factor is de Rham.*

Applying Theorem 3.10 to the tensor factor $V_{\mathbb{Q}_p}$ in Proposition 3.8, one obtains the following

Proposition 3.11. *Making an additional finite field extension $E' \subset E''$ if necessary, one has a further decomposition of $\text{Gal}_{E''}$ -representation:*

$$V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_{p^r}} \cong V_1 \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1, \sigma^{r-1}},$$

where $\text{Gal}_{E''}$ acts on \mathbb{Q}_{p^r} trivially and V_{1, σ^i} is the σ^i -conjugate of V_1 . Then V_1 is potentially crystalline.

Each σ -conjugate V_{1, σ^i} is isomorphic to V_1 as \mathbb{Q}_p -representation. Thus each tensor factor in the above decomposition is potentially crystalline as well.

Proof. Assume $r = 2$ for simplicity. The above tensor decomposition implies the tensor decomposition \mathbb{C}_p -representations:

$$V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong (V_1 \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p) \otimes_{\mathbb{C}_p} (V_{1, \sigma} \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p).$$

Since $V_{\mathbb{Q}_p}$ is crystalline, it is Hodge-Tate. It implies that the Sen's operator Θ_V of $V_{\mathbb{Q}_p}$ is diagonalizable (over \mathbb{C}_p). Let Θ_{V_1} be the Sen's operator of V_1 . It can be written naturally as $\Theta_1 \oplus \Theta_{1, \sigma}$ where Θ_1 is associated with $V_1 \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p$ and $\Theta_{1, \sigma}$ to $V_{1, \sigma} \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p$. Thus one has

$$\Theta_V = \Theta_1 \otimes id + id \otimes \Theta_{1, \sigma}.$$

It implies that Θ_1 and $\Theta_{1, \sigma}$ are diagonalizable. Now consider the eigenvalues of them. For that we use the relation between the Hodge-Tate cocharacter and the eigenvalues of the Sen's operator: they are related by the maps \log and \exp . Continue the argument about Hodge-Tate cocharacter in Proposition 3.8. So let $\{\tau = \tau_1, \tau_2\}$ be the $\text{Gal}_{\mathbb{Q}_p}$ -orbit of τ . We can assume that in the above decomposition the V_1 -factor corresponds to τ . It follows that the projection of μ_{HT} to the $V_{1, \sigma}$ -factor is trivial. This implies that the eigenvalues of $\Theta_{1, \sigma}$ are zero.

Particularly they are integral. So are those of Θ_1 . Hence Θ_{V_1} is diagonalizable with integral eigenvalues. So V_1 is Hodge-Tate, and by Theorem 3.10 it is de Rham. By the p -adic monodromy theorem, conjectured by Fontaine and firstly proved by Berger (see [1]), it is potentially log crystalline. One shows further that it is potentially crystalline. Let N_V (resp. N_{V_1}) be the monodromy operator of V (resp. V_1). Then one has the formulas:

$$N_{V_1} = N_1 + N_{1,\sigma}, \quad N_V = N_1 \otimes id + id \otimes N_{1,\sigma}.$$

Since V is crystalline, $N_V = 0$. It implies that $N_1 = N_{1,\sigma} = 0$. Hence $N_{V_1} = 0$ and V_1 is potentially crystalline. \square

3.3. Consequence on the possibilities of the Newton polygon. We show that the admissibility of a filtered ϕ -module associated with a crystalline representation yields a classification of the Newton polygons in the current case. The following functors were introduced by Fontaine in order to study a \mathbb{Q}_{p^r} -representation:

Definition 3.12. Let V be a \mathbb{Q}_{p^r} -representation. For each $0 \leq m \leq r-1$, one defines

$$D_{crys,r}^{(m)}(V) := (V \otimes_{\mathbb{Q}_{p^r}, \sigma^m} B_{crys})^{\text{Gal}_E}.$$

One defines the functors $\{D_{dR,r}^{(m)}(V)\}_{0 \leq m \leq r-1}$ by replacing B_{crys} with B_{dR} . The following lemma is obvious:

Lemma 3.13. *Let V be a \mathbb{Q}_{p^r} -representation. Then there is a natural isomorphism of Gal_E -representations:*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong \bigoplus_{m=1}^{r-1} V \otimes_{\mathbb{Q}_{p^r}, \sigma^m} \mathbb{Q}_{p^r}$$

By the lemma there is a natural direct decomposition:

$$V \otimes_{\mathbb{Q}_p} B_{crys} \cong V \otimes_{\mathbb{Q}_p} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_{p^r}} B_{crys}) \cong (V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} B_{crys} \cong \bigoplus_{m=0}^{r-1} V \otimes_{\mathbb{Q}_{p^r}, \sigma^m} B_{crys},$$

which implies in particular a direct decomposition of E_0 -vector spaces:

$$D_{crys}(V) = \bigoplus_{m=0}^{r-1} D_{crys,r}^{(m)}(V).$$

It is clear that V is crystalline iff $\dim_{E_0} D_{crys,r}^{(m)}(V) = \dim_{\mathbb{Q}_{p^r}} V$ for either m holds. Let V be a crystalline \mathbb{Q}_{p^r} -representation. Over $D_{crys}(V)$ there is a natural σ -linear map ϕ , and over $D_{dR}(V) = D_{crys}(V) \otimes_{E_0} E$ there is a natural filtration Fil . We want to study some properties of the restrictions of them to a direct factor.

Lemma 3.14. *The map ϕ permutes the direct factors $\{D_{crys,r}^{(m)}(V)\}$ cyclically. Consequently, one has the decomposition of ϕ^r -modules:*

$$(D_{crys,r}(V), \phi^r) = \bigoplus_{m=0}^{r-1} (D_{crys,r}^{(m)}(V), \phi^r|_{D_{crys,r}^{(m)}(V)}).$$

Moreover, each ϕ^r -submodule $(D_{crys,r}^{(m)}(V), \phi^r|_{D_{crys,r}^{(m)}(V)})$ has the same Newton slopes.

Proof. For $d = v \otimes_{\sigma^m} b \in D_{crys,r}^{(m)}(V)$, from the formula

$$\phi(d) = v \otimes_{\sigma^{m+1} \mod r} \phi(b),$$

we see that $\phi(d) \in D_{crys,r}^{(m+1 \bmod r)}(V)$. So ϕ permutes the direct factors in a cyclic way, and particularly ϕ^r is preserved under the direct decomposition. To show the last statement, we choose a finite extension E' of E such that $(f_{E'}, r) = 1$, where $f_{E'}$ denotes for the residue degree of E' . As a $\text{Gal}_{E'}$ -representation V is still crystalline, and it holds that

$$(D_{crys,E'}(V), \phi) = (D_{crys}(V) \otimes_{E_0} E'_0, \phi \otimes \sigma),$$

where $D_{crys,E'}(V)$ means $(V \otimes_{\mathbb{Q}_p} B_{crys})^{\text{Gal}_{E'}}$. There is a similar formula for $(D_{crys,E'}^{(m)}(V), \phi^r)$. So the Newton slopes do not change by replacing E with E' in question. By the choice of E' , one can write $af_{E'} = 1 + br$ for certain $a, b \in \mathbb{N}$. Since $\phi^{f_{E'}}$ is a linear isomorphism, $\phi^{af_{E'}}$ is also a linear isomorphism, which maps $D_{crys,r}^{(m)}(V)$ to $D_{crys,r}^{(m+1 \bmod r)}(V)$. Clearly, this is also an isomorphism as ϕ^r -modules. Hence the lemma follows. \square

As a consequence, one could define on the tensor product $\otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V)$ a ϕ -module structure: For a vector of form $v_0 \otimes \cdots \otimes v_{r-1}$, define

$$\phi_{ten}(v_0 \otimes \cdots \otimes v_{r-1}) := \phi(v_{r-1}) \otimes \phi(v_0) \otimes \cdots \otimes \phi(v_{r-2}).$$

It is easily seen that the σ^r -linear map ϕ_{ten}^r is the tensor product of $\phi^r|_{D_{crys,r}^{(m)}(V)}$ s. Next we consider the induced filtration on each direct factor $D_{dR,r}^{(m)}(V)$ from $D_{dR}(V)$:

$$Fil_m^i := Fil^i \cap D_{dR,r}^{(m)}(V).$$

As filtered modules it holds that

$$(D_{dR}(V), Fil) = \oplus_{m=0}^{r-1} (D_{dR,r}^{(m)}(V), Fil_m).$$

The tensor product $\otimes_{m=0}^{r-1} D_{dR,r}^{(m)}(V)$ is equipped with the filtration Fil_{ten} which is the tensor product of Fil_m s.

Proposition 3.15. *Let V_1 be a crystalline \mathbb{Q}_{p^r} -representation and V a \mathbb{Q}_p representation such that there is an isomorphism of \mathbb{Q}_{p^r} -representations*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}.$$

Then there is an isomorphism of filtered ϕ^r -modules:

$$D_{crys}(V) \cong \otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1).$$

Here the ϕ^r -structure on $D_{crys}(V)$ is the r -th power of the ϕ -structure on $D_{crys}(V)$ and the filtered ϕ^r structure on $\otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1)$ is the Fil_{ten} and ϕ_{ten}^r stated above.

Proof. Write $\tilde{V} = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}$. First of all, one has a natural isomorphism of filtered ϕ^r -modules

$$D_{crys}(V) = (V \otimes_{\mathbb{Q}_p} B_{crys})^{\text{Gal}_E} \cong ((V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} B_{crys})^{\text{Gal}_E} = D_{crys,r}^{(0)}(\tilde{V}).$$

Secondly, the tensor decomposition of \tilde{V} in the condition implies that there is a natural isomorphism of filtered ϕ^r -modules

$$D_{crys,r}^{(0)}(\tilde{V}) \cong \otimes_{m=0}^{r-1} D_{crys,r}^{(0)}(V_{1,\sigma^m}).$$

Because it holds that

$$D_{crys,r}^{(0)}(V_{1,\sigma^m}) = ((V_1 \otimes_{\mathbb{Q}_{p^r}, \sigma^m} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} B_{crys})^{\text{Gal}_E} \cong (V_1 \otimes_{\mathbb{Q}_{p^r}, \sigma^m} B_{crys})^{\text{Gal}_E} = D_{crys,r}^{(m)}(V_1),$$

one has a natural isomorphism of filtered ϕ^r -modules

$$(D_{crys,r}^{(m)}(V_1), \phi^r, \text{Fil}_m) \cong (D_{crys,r}^{(0)}(V_{1,\sigma^m}), \phi^r, \text{Fil}_0).$$

Combining these we have shown that there is a natural isomorphism of filtered ϕ^r -modules:

$$D_{crys}(V) \cong \otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1).$$

□

One can improve the above result by showing that it is even an isomorphism of filtered ϕ -modules. Here we equip the filtration on $\otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1)$ with the previous one, and the ϕ -structure with ϕ_{ten} .

Proposition 3.16. *Let V_1 and V be as in Proposition 3.15. Then there is a natural isomorphism of filtered ϕ -modules:*

$$D_{crys}(V) \cong \otimes_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1).$$

Proof. Notation as above. In the following we denote the tensor product of V_{1,σ^i} s as \mathbb{Q}_{p^r} -representations (resp. \mathbb{Q}_p -representations) by $\bigotimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^i}$, (resp. $\bigotimes_{\mathbb{Q}_p} V_{1,\sigma^i}$). One notes that there is an inclusion of \mathbb{Q}_p -representations:

$$V \subset \tilde{V} \cong \bigotimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^i} \subset \bigotimes_{\mathbb{Q}_p} V_{1,\sigma^i} \cong V_1^{\otimes r}.$$

It implies that $D_{crys}(V)$ is naturally a filtered ϕ submodule of

$$D_{crys}(V_1^{\otimes r}) \cong D_{crys}(V_1)^{\otimes r}$$

through the above isomorphisms. Applying the functor D_{crys} to the second inclusion in the above expression, one knows that the image of the inclusion

$$D_{crys}(\bigotimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^i}) \subset (\oplus_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1))^{\otimes r}$$

is invariant under the map $\phi^{\otimes r}$. For an r -tuple (i_0, \dots, i_{r-1}) , where $0 \leq i_0, \dots, i_{r-1} \leq r-1$, we set

$$\bigotimes_{(i_0, \dots, i_{r-1})} D_{crys,r}(V_1) := D_{crys,r}^{(i_0)}(V_1) \otimes D_{crys,r}^{(i_1)}(V_1) \otimes \dots \otimes D_{crys,r}^{(i_{r-1})}(V_1).$$

Now as

$$D_{crys}(\bigotimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^i}) \supset D_{crys,r}^{(0)}(\bigotimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^i}) = \bigotimes_{(0, \dots, r-1)} D_{crys,r}(V_1)$$

canonically, the corresponding ϕ -submodule in $(\oplus_{m=0}^{r-1} D_{crys,r}^{(m)}(V_1))^{\otimes r}$ is

$$\bigotimes_{(0,1,\dots,r-2,r-1)} D_{crys,r}(V_1) \oplus \bigotimes_{(1,2,\dots,r-1,0)} D_{crys,r}(V_1) \oplus \dots \oplus \bigotimes_{(r-1,0,\dots,r-3,r-2)} D_{crys,r}(V_1).$$

Note that ϕ permutes the direct factors cyclically in the decomposition. Also since as \mathbb{Q}_p -representations one has isomorphisms

$$\begin{aligned}\tilde{V} \otimes_{\mathbb{Q}_p} B_{crys} &\cong V \otimes_{\mathbb{Q}_p} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} B_{crys} \\ &\cong (V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_{p^r}} B_{crys})^{\oplus r} \\ &\simeq (V \otimes_{\mathbb{Q}_p} B_{crys})^{\oplus r},\end{aligned}$$

it follows that $D_{crys}(\tilde{V}) \cong D_{crys}(V)^{\oplus r}$. The first inclusion $V \subset \tilde{V}$ realizes $D_{crys}(V)$ as one of factors in $D_{crys}(\tilde{V})$. This means that the filtered ϕ -module

$$\bigotimes_{(0,1,\dots,r-2,r-1)} D_{crys,r}(V_1) \oplus \bigotimes_{(1,2,\dots,r-1,0)} D_{crys,r}(V_1) \oplus \dots \oplus \bigotimes_{(r-1,0,\dots,r-3,r-2)} D_{crys,r}(V_1)$$

is in fact isomorphic to the direct sum of r copies of the filtered ϕ -module $D_{crys}(V)$. For that, one takes $r = 2$ as an example (for a general r the isomorphism is similar): One has an obvious isomorphism of E_0 -vector spaces:

$$\Delta : (D_{crys,2}^{(0)}(V_1) \otimes D_{crys,2}^{(1)}(V_1))^{\oplus 2} \cong \bigotimes_{(0,1)} D_{crys,2}(V_1) \oplus \bigotimes_{(1,0)} D_{crys,2}(V_1),$$

where Δ on the first factor is the identity map and on the second factor is to change the positions of two tensor factors. One verifies easily that under the isomorphism Δ^{-1} , the above filtered ϕ -module structure is isomorphic to the direct sum of two copies of the filtered ϕ -module structure on $D_{crys,2}^{(0)}(V_1) \otimes D_{crys,2}^{(1)}(V_1)$ with the tensor product of filtrations on factors and ϕ -structure the ϕ_{ten} . This shows the proposition. \square

In the previous propositions we consider the case where V is polarisable and of Hodge-Tate weights $\{0, 1\}$. Here V is polarisable means that there is a perfect Gal_E -pairing $V \otimes V \rightarrow \mathbb{Q}_p(-1)$. This condition implies that if λ is a Newton (resp. Hodge) slopes of V , then $1 - \lambda$ is also a Newton (resp. Hodge) slopes of V with the *same* multiplicity.

Proposition 3.17. *Let V be a polarisable crystalline representation with Hodge-Tate weights $0, 1$. If there exists a two dimensional crystalline \mathbb{Q}_{p^r} -representation V_1 such that*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \dots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}$$

holds, then it holds that

- (i) *the Hodge slopes of V_1 is $\{2r - 1 \times 0, 1 \times 1\}$,*
- (ii) *the Newton slopes of V_1 is either $\{2r \times \frac{1}{2r}\}$ or $\{r \times 0, r \times \frac{1}{r}\}$.*

Consequently, there are only two possible Newton slopes for V : $\{2^r \times \frac{1}{2}\}$ or $\{1 \times 0, \dots, \binom{r}{i} \times \frac{i}{r}, \dots, 1 \times 1\}$.

Proof. Since the Hodge slopes of V are $\{n \times 0, n \times 1\}$, by Proposition 3.15, there exists a unique factor $D_{crys,r}^{(i)}(V_1)$ with two distinct Hodge slopes $\{0, 1\}$ and the other factors have all Hodge slopes zero. Without loss of generality one can assume that $D_{crys,r}^{(0)}(V_1)$ has Hodge slopes $\{1 \times 0, 1 \times 1\}$ (and any other factor $\{2 \times 0\}$). Summing up the Hodge slopes of all factors, one obtains the Hodge polygon of $D_{crys}(V_1)$ as claimed. By the admissibility of the filtered ϕ -module structure on

$D_{crys}(V_1)$, one finds that its Newton slopes must be of form $\{m_1 \times 0, m_2 \times \lambda\}$ where $m_1 + m_2 = 2r$ holds, and $\lambda \in \mathbb{Q}$ satisfies $\lambda m_2 = 1$. By Lemma 3.14, one finds that $r|m_i$, $i = 1, 2$. So $\frac{m_1}{r} + \frac{m_2}{r} = 2$ and $m_2 \neq 0$. There are only two possible cases:

Case 1: $m_1 = 0$. It implies that $m_2 = 2r$ and $\lambda = \frac{1}{2r}$.

Case 2: $m_1 \neq 0$. It implies that $m_1 = m_2 = r$ and $\lambda = \frac{1}{r}$. \square

Now we can prove the main result of the first part:

Theorem 3.18. *Notation as above. Then there are two possible Newton polygons in $M_0(\bar{k})$. Precisely it is either $\{2^{d+\epsilon(D)} \times \frac{1}{2}\}$ or*

$$\{2^{d-r+\epsilon(D)} \times 0, \dots, 2^{d-r+\epsilon(D)} \cdot \binom{r}{i} \times \frac{i}{r}, \dots, 2^{d-r+\epsilon(D)} \times 1\}.$$

Proof. Let $x_0 \in M_0(k)$, A_{x_0} and A_x as in §3.1. By the p -adic Hodge theory, the Newton polygon of A_{x_0} is that of the filtered ϕ -module $D_{crys}(H_{et}^1(\bar{A}_{x^0}, \mathbb{Q}_p))$. For a suitable large finite extension E of \mathbb{Q}_p , we can assume the tensor decomposition of the Gal_E -module $H_{et}^1(\bar{A}_{x^0}, \mathbb{Q}_p)$ as given in Corollary 3.7. Now Propositions 3.8 and 3.11 imply that we can even further assume that each tensor factor in the tensor decomposition is crystalline. An unramified factor contributes only to a multiplicity in the Newton polygon. Hence the theorem follows directly from Proposition 3.17. \square

Remark 3.19. The above result generalizes a result of Noot in [28], [29]. Our method differs from his. In the following we would like to discuss the existence of each Newton polygon in the classification. To this end, it suffices to realize that the method of Noot for the original example of Mumford (see §3-5 [29]) can be generalized directly: Noot studied the reductions of CM points of a Mumford's family. The set of CM points can be divided into two types: Let $F \subset J$ be a maximal subfield of D . Then J can either be written in a form $N \otimes_{\mathbb{Q}} F$ (N is necessary an imaginary quadratic extension of \mathbb{Q}) or not in such a form. To our purpose one finds that the second case generalizes, and the resulting generalization gives the necessary existence result. More precisely, Proposition 5.2 in loc. cit. provides the maximal subfields in D of the second type with the following freedom: Let \mathfrak{p} be a prime of F over p . Then J can be so chosen that \mathfrak{p} is split or inert in J . Secondly, Lemma 3.5 and Proposition 3.7 in loc. cit. work verbatim for a general D except in the case of $\epsilon(D) = 1$, one adds the multiplicity two to the constructions appeared therein. This step gives us an isogeny class of CM abelian varieties which appear as \mathbb{Q} -points of M_K , and also as \mathbb{Z}_p -points of M and hence in $M_0(\bar{k})$. Finally the proof of Proposition 4.4 in loc. cit., namely the method of computing the Newton polygon for a CM abelian variety modulo p , works in general. Thus one can also conclude the existence result for the general case.

4. A FORMAL TENSOR DECOMPOSITION OF THE UNIVERSAL FILTERED DIEUDONNÉ MODULE

Recall that the universal filtered Dieudonné module $(H, F, \nabla, \phi) \in MF_{[0,1]}^\nabla(M)$. Let x_0 be a k -rational point of M and \hat{M}_{x_0} the completion of M at x_0 . The aim of this section is to show a tensor decomposition of the restriction of (H, F, ∇, ϕ) to the formal neighborhood of x_0 . This is an application of the deformation theory of p -divisible groups with Tate cycles due to Faltings (see §7 [13], see also §4 [22] and §1.5 [18]) and the theory of integral canonical models of Kisin (see [18]). Let $F_p \subset L \subset E$ be a finite extension and x^0 an E -rational point of M which specializes into x_0 . Corollary 3.7 gives a decomposition of $H_{\mathbb{Z}_p}$ into tensor product of Gal_E -lattices. By Proposition 3.11 and Proposition 3.17, $V_1 \subset V_1 \otimes \mathbb{Q}_p$ is a Gal_E -lattice of a two dimensional potentially crystalline \mathbb{Q}_{p^r} -representation with Hodge-Tate weights $\{2r - 1 \times 0, 1 \times 1\}$. By making a possible finite extension of E , we can assume that $V_1 \otimes \mathbb{Q}_p$ is already crystalline as Gal_E -module.

4.1. Drinfel'd's \mathcal{O}_p -divisible module and versal deformation. Recall the following notion of \mathcal{O}_p -divisible modules due to Drinfel'd (see Appendix [5]):

Definition 4.1. Let $S = \text{Spec} R$ be an \mathcal{O}_p -scheme. An \mathcal{O}_p -divisible module over S is a pair (G, f) consisting of a p -divisible group G over S and an action of \mathcal{O}_p on G :

$$f : \mathcal{O}_p \rightarrow \text{End}(G)$$

satisfying

- (a) $f(1)$ is the identity,
- (b) G^0 (the connected part of G) is of dimension 1,
- (c) the derivation of f , $f' : \mathcal{O}_p \rightarrow \text{End}(\text{Lie}(G)) = R$ coincides with the structural morphism $\mathcal{O}_p \rightarrow R$.

By the fundamental theorem of Breuil (see Corollary 3.2.4, Theorem 3.2.5 [4]), the Gal_E -lattice V_1 corresponds to a p -divisible group B over \mathcal{O}_E .

Lemma 4.2. *The corresponding p -divisible group B is an \mathcal{O}_p -divisible module over \mathcal{O}_E of height 2.*

Proof. By construction, one has the inclusion

$$\mathbb{Z}_{p^r} \subset \text{End}_{\text{Gal}_E}(V_1).$$

Since the functor of Breuil is an anti-equivalence of categories, one obtains an inclusion as well

$$\mathcal{O}_p \cong \mathbb{Z}_{p^r} \subset \text{End}(B).$$

The condition (a) is obvious. The condition (b) on the dimension of G and the assertion on the height of B follow from the Hodge-Tate weights of $V_1 \otimes \mathbb{Q}_p$ given in Proposition 3.17. By taking the derivation of the inclusion, one obtains an inclusion of \mathbb{Z}_p -algebras $\mathcal{O}_p \subset \mathcal{O}_E$, which ought to be the structural morphism by the naturalness of the functor. \square

Let M_B be the filtered Dieudonné module associated with B . Denote by L_B the previous Gal_E -lattice V_1 . Fix a generator s of \mathbb{Z}_{p^r} as \mathbb{Z}_p -algebra. The image of s in $\text{End}_{\text{Gal}_E}(L_B) \subset \text{End}_{\mathbb{Z}_p}(L_B) \subset L_B^{\otimes}$ is an étale Tate cycle of L_B and will be denoted by $s_{B,et}$. By the p -adic comparison, one has a natural isomorphism respecting Gal_E -action, filtrations and ϕ s:

$$L_B \otimes_{\mathbb{Z}_p} B_{crys} \cong M_B \otimes_{W(k)} B_{crys}.$$

By one of the main technical results of [18] (see Proposition 0.2 in loc. cit.), the crystalline Tate cycle s_B , which corresponds to $s_{B,et} \otimes 1 \in L_B \otimes_{\mathbb{Z}_p} B_{crys}$ in the above comparison, lies actually in M_B^{\otimes} . Let $G_B \subset \text{GL}_{\mathbb{Z}_p}(L_B)$ (resp. $\mathcal{G}_B \subset \text{GL}_{W(k)}(M_B)$) be the subgroup defined by $s_{B,et}$ (resp. s_B). By Corollary 1.4.3 (4) in loc. cit., the filtration $Fil^1 \otimes k$ on $M_B \otimes k$ is $\mathcal{G}_B \otimes_{W(k)} k$ -split. Choose a cocharacter $\mu_0 : \mathbb{G}_m \rightarrow \mathcal{G}_B \otimes k$ inducing this filtration and further choose a cocharacter $\mu : \mathbb{G}_m \rightarrow \mathcal{G}_B \subset \text{GL}(M_B)$ lifting μ_0 (see 1.5.4 loc. cit.). The cocharacter μ defines the opposite unipotent subgroups:

$$\begin{array}{ccc} U_{\mathcal{G}_B} & \xrightarrow{\subset} & \mathcal{G}_B \\ \cap \downarrow & & \downarrow \cap \\ U_B & \xrightarrow{\subset} & \text{GL}(M_B) \end{array}$$

By construction, one has $U_{\mathcal{G}_B} = U_B \cap \mathcal{G}_B$. Let $\hat{U}_{\mathcal{G}_B}$ (resp. \hat{U}_B) be the completion of $U_{\mathcal{G}_B}$ at the identity section of $U_{\mathcal{G}_B}$ (resp. U_B), whose corresponding complete local rings are denoted by $R_{\mathcal{G}_B}$ and R respectively. The filtration on M_B defined by μ corresponds to a p -divisible group B' over $W(k)$ whose closed fiber $B' \otimes k$ is isomorphic to $B \otimes k$ as p -divisible group over k . For a later use we denote this μ also by $\mu_{B'}$. Write $(M'_B, Fil^1_{M'_B}, \phi_{M'_B})$ to be the tuple defined by the filtered crystal structure on M'_B , and fix $\phi_{\hat{U}_{\mathcal{G}_B}} : R_{\mathcal{G}_B} \rightarrow R_{\mathcal{G}_B}$ a lifting of the absolute Frobenius. Following Faltings Remarks §7 [13] (see also §4 [22] and §1.5 [18]), one defines a filtered F -crystal over $\hat{U}_{\mathcal{G}_B}$ by the tuple

$$(\mathcal{N}_B = M'_B \otimes_{W(k)} R_{\mathcal{G}_B}, Fil^1_{\mathcal{N}_B} = Fil^1_{M'_B} \otimes_{W(k)} R_{\mathcal{G}_B}, \phi_{\mathcal{N}_B} = u \circ (\phi_{M'_B} \otimes \phi_{\hat{U}_{\mathcal{G}_B}})),$$

where $u \in U_{\mathcal{G}_B}(R_{\mathcal{G}_B})$ is the tautological $R_{\mathcal{G}_B}$ -point of $U_{\mathcal{G}_B}$. Then by Faltings loc. cit., there is a unique integrable connection $\nabla_{\mathcal{N}_B}$ over \mathcal{N}_B such that the four tuple $(\mathcal{N}_B, Fil^1_{\mathcal{N}_B}, \phi_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B})$ defines an object in $MF^{\nabla}_{[0,1]}(\hat{U}_{\mathcal{G}_B})$. By Faltings Theorem 7.1 [12], there is a p -divisible group \mathcal{B} over $R_{\mathcal{G}_B}$, unique up to isomorphism, such that the attached filtered Dieudonné module to \mathcal{B} is isomorphic to the above four tuple. If we replace everything of $U_{\mathcal{G}_B}$ with that of U_B , the above discussion gives then a versal deformation of $B \otimes k$ over \hat{U}_B , which by abuse of notation is denoted again by \mathcal{B} . In this context, the sublocus $\text{Spf}(R_{\mathcal{G}_B}) \hookrightarrow \text{Spf}(R)$ has an interpretation as the versal deformation respecting the Tate cycles which are stabilized by \mathcal{G}_B (see §7 [13], Proposition 4.9 [22] and Corollary 1.5.5 [18]). By Corollary 1.5.11 [18], B is isomorphic to the pull-back of \mathcal{B} along a $W(k)$ -algebra morphism $R_{\mathcal{G}_B} \rightarrow \mathcal{O}_E$.

We proceed to study the natural \mathcal{G}_B -action on M_B via the inclusion $\mathcal{G}_B \subset \mathrm{GL}_{W(k)}(M_B)$. Recall that we have fixed an element $s \in \mathbb{Z}_{p^r}$. Let $\{s_i := s^{\sigma^i}\}_{0 \leq i \leq r-1} \subset \mathbb{Z}_{p^r}$ be the Galois conjugates of s . The minimal polynomial of $s_B \in \mathrm{End}_{W(k)}(M_B)$ is that of $s \in \mathbb{Q}_{p^r}$ over \mathbb{Q}_p . As $\mathbb{Z}_{p^r} \subset W(k)$, the minimal polynomial of s_B splits into linear factors and one has then the relation in $\mathrm{End}_{W(k)}(M_B)$:

$$(s_B - s_0) \cdots (s_B - s_{r-1}) = 0.$$

Let $M_i \subset M_B$ be the eigenspace of s_B corresponding to the eigenvalue s_i . Recall that we have shown in §3 a direct sum decomposition of $M_B \otimes \mathrm{Frac}(W(k)) = D_{\mathrm{crys}}(L_B \otimes \mathbb{Q}_p)$ by using the functors $D_{\mathrm{crys},r}^{(i)}$ of Fontaine. Now we have the following

Lemma 4.3. *The eigen decomposition $M_B = \bigoplus_{i=0}^{r-1} M_i$ is a lattice decomposition of*

$$M_B \otimes \mathrm{Frac}(W(k)) = \bigoplus_{i=0}^{r-1} D_{\mathrm{crys},r}^{(i)}(L_B \otimes \mathbb{Q}_p).$$

Namely, M_i is a lattice in $D_{\mathrm{crys},r}^{(i)}(L_B \otimes \mathbb{Q}_p)$ for each i .

Proof. In the comparison isomorphism $L_B \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}} \cong M_B \otimes_{W(k)} B_{\mathrm{crys}}$, the endomorphism $s_{B,\mathrm{et}} \otimes 1$ corresponds to $s_B \otimes 1$. Also the endomorphisms commute with the Gal_E -actions on both sides. The endomorphism $s_{B,\mathrm{et}} \otimes 1$ on $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ decomposes into eigenspaces and so we obtain a $\mathbb{Z}_{p^r}[\mathrm{Gal}_E]$ -module decomposition:

$$L_B \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}} = (L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}) \otimes_{\mathbb{Z}_{p^r}} B_{\mathrm{crys}} = \bigoplus_{i=0}^{r-1} L_i \otimes_{\mathbb{Z}_{p^r}} B_{\mathrm{crys}},$$

where L_i is the \mathbb{Z}_{p^r} -submodule of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ corresponding to the eigenvalue s_i . Under the comparison isomorphism it corresponds to the decomposition of $W(k)[\mathrm{Gal}_E]$ -module:

$$M_B \otimes_{W(k)} B_{\mathrm{crys}} = \bigoplus_{i=0}^{r-1} M_i \otimes_{W(k)} B_{\mathrm{crys}}.$$

Taking the Gal_E -invariants, we obtain

$$(L_i \otimes_{\mathbb{Z}_{p^r}} B_{\mathrm{crys}})^{\mathrm{Gal}_E} = M_B \otimes_{W(k)} \mathrm{Frac}(W(k)).$$

Finally one notices that the two indexed sets of Gal_E -modules $\{L_i \otimes_{\mathbb{Z}_{p^r}} B_{\mathrm{crys}}\}_{0 \leq i \leq r-1}$ and $\{L_B[\frac{1}{p}] \otimes_{\mathbb{Q}_{p^r}, \sigma^i} B_{\mathrm{crys}}\}_{0 \leq i \leq r-1}$ are actually equal. The lemma follows. \square

Corollary 4.4. *The tensor product $\bigotimes_{i=0}^{r-1} M_i$ is a lattice of the admissible filtered ϕ -module $\bigotimes_{i=0}^{r-1} D_{\mathrm{crys},r}^{(i)}(L_B \otimes \mathbb{Q}_p)$ in Proposition 3.16.*

Proof. The filtration Fil_B^1 on $M_B \otimes_{W(k)} \mathcal{O}_E$ is filtered free and restricts to the filtration Fil_i^1 on each direct factor $M_i \otimes \mathcal{O}_E$. Also one sees from the proof of Proposition 3.17 that there is a unique factor M_i with nontrivial Fil_i^1 . Since $\phi_{M'_B}$ is σ -linear, it permutes the eigen factors $\{M_i\}_{0 \leq i \leq r-1}$ cyclically. \square

For a later use, we denote it by $(\bigotimes_{i=0}^{r-1} M_i, \mathrm{Fil}_{\mathrm{ten}}^1, \phi_{\mathrm{ten}})$.

Lemma 4.5. *The eigen decomposition of M_B is also a decomposition as \mathcal{G}_B -module. In fact, the $W(k)$ -group \mathcal{G}_B is naturally isomorphic to $\prod_{i=0}^{r-1} \mathrm{GL}_2(W(k))$, and the \mathcal{G}_B -module M_B is isomorphic to the $\prod_{i=0}^{r-1} \mathrm{GL}_2(W(k))$ -module $\bigoplus_{i=0}^{r-1} (W(k)^{\oplus 2})_i$,*

in which the i -th factor $(W(k)^{\oplus 2})_i$ is the tensor product of the standard representation of the i -th factor $\mathrm{GL}_2(W(k))$ and the trivial representations of the j -th factors with $j \neq i$.

Proof. Because the \mathcal{G}_B -action on M_B commutes with the s_B -action by definition, the eigen decomposition of M_B with respect to s_B is preserved by the \mathcal{G}_B -action. This can be seen more clearly if we go to the étale side: First of all, it is easy to see that the commutant subalgebra of

$$\mathbb{Z}_{p^r} \subset \mathrm{End}_{\mathbb{Z}_p}(L_B) \cong M_{2r}(\mathbb{Z}_p)$$

is $\mathrm{End}_{\mathbb{Z}_{p^r}}(L_B) \cong M_2(\mathbb{Z}_{p^r})$. So the group $G_B \subset \mathrm{GL}_{\mathbb{Z}_p}(L_B) \cong \mathrm{GL}_{2r}(\mathbb{Z}_p)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z}_{p^r})$, and particularly it is connected. Next, by Corollary 1.4.3 (3) [18], there is a $W(k)$ -linear isomorphism $L_B \otimes_{\mathbb{Z}_p} W(k) \cong M_B$ which induces an isomorphism

$$G_B \times_{\mathbb{Z}_p} W(k) \cong \mathcal{G}_B.$$

As $\mathbb{Z}_{p^r} \subset W(k)$, one has isomorphisms

$$\mathcal{G}_B \cong \mathrm{GL}_2(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} W(k)) \cong \mathrm{GL}_2\left(\prod_{i=0}^{r-1} W(k)\right) = \prod_{i=0}^{r-1} \mathrm{GL}_2(W(k)).$$

Under the above isomorphism, the \mathcal{G}_B -module M_B is isomorphic to the G_B -module L_B tensoring with $W(k)$. Moreover the isomorphism preserves the eigen decompositions of both sides. As said above, the G_B -action on L_B is isomorphic to the standard representation of $\mathrm{GL}_2(\mathbb{Z}_{p^r})$ on $\mathbb{Z}_{p^r}^{\oplus 2}$, which is considered as a \mathbb{Z}_p -group acting on a \mathbb{Z}_p -module by restriction of scalar. Thus, the action tensoring with \mathbb{Z}_{p^r} , that is the standard $\mathrm{GL}_2(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})$ -action on $(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^{\oplus 2}$, splits: Write

$$\mathrm{GL}_2(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}) \cong \prod_{i=0}^{r-1} \mathrm{GL}_2(\mathbb{Z}_{p^r}), \quad g \mapsto (g_0, \dots, g_{r-1}),$$

and

$$(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^{\oplus 2} \cong \prod_{i=0}^{r-1} \mathbb{Z}_{p^r}^{\oplus 2}, \quad v \mapsto (v_0, \dots, v_{r-1}).$$

Then $g(v)$ is mapped to $(g_0 v_0, \dots, g_{r-1} v_{r-1})$. So does the action after tensoring with the larger ring $W(k)$. Hence the lemma follows. \square

Proposition 4.6. *The sublocus $\hat{U}_{\mathcal{G}_B}$ is the versal deformation of the p -divisible group $B \otimes k$ as \mathcal{O}_p -divisible module.*

Proof. Calculate first the dimension of $\hat{U}_{\mathcal{G}_B}$: It is equal to $\mathrm{rank}_{W(k)} \frac{\mathfrak{g}}{\mathrm{Fil}^0 \mathfrak{g}}$, where \mathfrak{g} is the Lie algebra of \mathcal{G}_B and the filtration is the restriction of the tensor filtration on $\mathrm{End}_{W(k)}(M_B) = M_B^* \otimes M_B$ via the inclusion $\mathfrak{g} \subset \mathrm{End}_{W(k)}(M_B)$. We claim that it is one dimensional. By the discussion on the filtration in the proof of Corollary 4.4 and Lemma 4.5, one has an isomorphism of Lie algebras over $W(k)$

$$\mathfrak{g} \cong \bigoplus_{i=0}^{r-1} \mathfrak{gl}_2,$$

such that there is a unique factor with the nontrivial induced filtration through the isomorphism. This shows the claim. Now let $\tilde{B} \rightarrow Z$ be the versal deformation

of $B \otimes k$ as $\mathcal{O}_{\mathfrak{p}}$ -divisible modules. Thus one has a map $\tilde{f} : \mathcal{O}_{\mathfrak{p}} \rightarrow \text{End}(\tilde{B})$ which makes the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{p}} & \xrightarrow{\tilde{f}} & \text{End}(\tilde{B}) \\ & \searrow f & \downarrow \otimes k \\ & & \text{End}(B) \end{array}$$

Let $s_{\text{cycle}} \in \text{End}(B)$ and $\tilde{s}_{\text{cycle}} \in \text{End}(\tilde{B})$ be the images of $s \in \mathcal{O}_{\mathfrak{p}}$ in the endomorphism \mathbb{Z}_p -algebras. The element s_{cycle} corresponds to s_B under the Dieudonné functor, and by Faltings Theorem 7.1 [12], the corresponding element \tilde{s}_B to \tilde{s}_{cycle} , as an endomorphism of the filtered Dieudonné crystal attached to \tilde{B} , is a crystalline Tate cycle and is the parallel continuation of s_B over Z . By the universal property Proposition 4.9 [22], the inclusion $Z \subset \hat{U}_B$ factors

$$Z \subset \hat{U}_{\mathcal{G}_B} \subset \hat{U}_B.$$

As both Z and $\hat{U}_{\mathcal{G}_B}$ are formally smooth of dimension one, $Z = \hat{U}_{\mathcal{G}_B}$. \square

Let $(\mathcal{N}_B, \text{Fil}_{\mathcal{N}_B}^1, \phi_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B})$ be the universal filtered Dieudonné module attached to \mathcal{B} over $\hat{U}_{\mathcal{G}_B}$. For $0 \leq i \leq r-1$ put $\mathcal{N}_i = M_i \otimes R_{\mathcal{G}_B}$. Then the Tate cycle $s_B \in \text{End}_{R_{\mathcal{G}_B}}(\mathcal{N}_B)$ induces the eigen decomposition

$$(\mathcal{N}_B, \text{Fil}_{\mathcal{N}_B}^1, \nabla_{\mathcal{N}_B}) = \bigoplus_{i=0}^{r-1} (\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \nabla_{\mathcal{N}_i}),$$

where $\text{Fil}_{\mathcal{N}_i}^1$ (resp. $\nabla_{\mathcal{N}_i}$) is the restriction of $\text{Fil}_{\mathcal{N}_B}^1$ (resp. $\nabla_{\mathcal{N}_B}$) to \mathcal{N}_i . However the eigen decomposition is not preserved by $\phi_{\mathcal{N}_B}$: Recall that $\phi_{\mathcal{N}_B} = u \circ (\phi_{M'_B} \otimes \phi_{\hat{U}_{\mathcal{G}_B}})$. As $U_{\mathcal{G}_B} \subset \mathcal{G}_B$, by Lemma 4.5 u preserves the eigen decomposition. So $\phi_{\mathcal{N}_B}$ permutes the factors in the eigen decomposition in a cyclic way. In order to state the following decomposition result, we need to introduce the category $MF_{\text{big},r}^{\nabla}$, which is analogous to the category MF_{big}^{∇} introduced by Faltings (see c)-d) Ch II [12]). The category $MF_{\text{big},r}^{\nabla}(R_{\mathcal{G}_B})$ consists of four tuples $(N, \text{Fil}, \phi_r, \nabla)$: N is a free $R_{\mathcal{G}_B}$ -module, Fil a sequence of $R_{\mathcal{G}_B}$ -submodules with $\text{Gr}_{\text{Fil}}(N)$ torsion free,

$$\phi_r : N \otimes_{R_{\mathcal{G}_B}, \phi_{\hat{U}_{\mathcal{G}_B}}^r} R_{\mathcal{G}_B} \rightarrow N$$

satisfies the divisibility condition $\phi_r(\text{Fil}^i) \subset p^i N$, and ∇ an integrable connection satisfying the Griffiths transversality and finally ϕ_r is parallel with respect to ∇ . Summarizing the above discussions, we have shown the

Proposition 4.7. *The object $(\mathcal{N}_B, \text{Fil}_{\mathcal{N}_B}^1, \nabla_{\mathcal{N}_B})$ has a decomposition*

$$(\mathcal{N}_B, \text{Fil}_{\mathcal{N}_B}^1, \nabla_{\mathcal{N}_B}) = \bigoplus_{i=0}^{r-1} (\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \nabla_{\mathcal{N}_i}),$$

such that $\phi_{\mathcal{N}_B}$ permutes the factors cyclically. Consequently, one has a direct sum decomposition in the category $MF_{\text{big},r}^{\nabla}(\hat{U}_{\mathcal{G}_B})$:

$$(\mathcal{N}_B, \text{Fil}_{\mathcal{N}_B}^1, \phi_{\mathcal{N}_B}^r, \nabla_{\mathcal{N}_B}) = \bigoplus_{i=0}^{r-1} (\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \phi_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}),$$

where $\phi_{\mathcal{N}_i}$ is the restriction of $\phi_{\mathcal{N}_B}$ to \mathcal{N}_i .

As a consequence, one can define an object in $MF_{[0,1]}^\nabla(\hat{U}_{\mathcal{G}_B})$ by equipping the tensor product $\otimes_{i=0}^{r-1}(\mathcal{N}_B, \text{Fil}_{\mathcal{N}_B}^1, \nabla_{\mathcal{N}_B})$ with the Frobenius ϕ_{ten} , a construction mimic to Corollary 4.4.

4.2. Tensor decomposition of the universal filtered Dieudonné module over a formal neighborhood. Notation as the introductory part of the section. Let A be the abelian scheme over \mathcal{O}_E with the closed fiber (resp. generic fiber) A_0 (resp. A^0) given by x_0 (resp x^0). Put $L_A = H_{\mathbb{Z}_p} = H_{et}^1(\bar{A}^0, \mathbb{Z}_p)$. For simplicity of notation, we use the same letters A etc. to mean the associated p -divisible groups. Recall that Corollary 3.7 gives a Gal_E -lattice decomposition:

$$L_A = (V_{\mathbb{Z}_p} \otimes U_{\mathbb{Z}_p})^{\oplus 2^{\epsilon(D)}}.$$

Let A_1 and A_2 be the two p -divisible groups over \mathcal{O}_E corresponding to the lattice $V_{\mathbb{Z}_p}$ and $U_{\mathbb{Z}_p}$ respectively by the theorem of Breuil (see [3]-[4]). Write $L_{A_1} = V_{\mathbb{Z}_p}$, $L_{A_2} = U_{\mathbb{Z}_p}$. Let $(M_A, \text{Fil}_A^1, \phi_A)$ be the filtered Dieudonné module attached to A . Similar notations for A_1 and A_2 .

Proposition 4.8. *One has a natural isomorphism of filtered ϕ -modules:*

$$(M_A, \text{Fil}_A^1, \phi_A) \cong [(M_{A_1}, \text{Fil}_{A_1}^1, \phi_{A_1}) \otimes (M_{A_2}, \text{Fil}_{A_2}^1, \phi_{A_2})]^{\oplus 2^{\epsilon(D)}},$$

where the factor $(M_{A_1}, \text{Fil}_{A_1}^1, \phi_{A_1})$ is naturally isomorphic to $(\otimes_{i=0}^{r-1} M_i, \text{Fil}_{ten}^1, \phi_{ten})$ in Corollary 4.4, and the factor $(M_{A_2}, \text{Fil}_{A_2}^1, \phi_{A_2})$ is a unit crystal.

Proof. We have shown the above isomorphisms after inverting p of both sides: The first isomorphism is a consequence of Propositions 3.8, 3.11 as the functor D_{crys} commutes with tensor product. The second isomorphism is Proposition 3.16. Also since $U_{\mathbb{Q}_p}$ is an unramified Gal_E -representation, (M_{A_2}, ϕ_{A_2}) is a unit crystal and the filtration $\text{Fil}_{A_2}^1$ is trivial. To show the isomorphisms hold without inverting p , we shall apply the theory of \mathfrak{S} -modules of Kisin developed in [17] and §1.2, 1.4 [18]. Consider the first isomorphism: Apply first the functor \mathfrak{M} to the Gal_E -lattice decomposition

$$L_A = (L_{A_1} \oplus L_{A_2})^{\oplus 2^{\epsilon(D)}}.$$

From the proof of Theorem 1.2.2 [18] (see 1.2.2 loc. cit.), one sees that the functor \mathfrak{M} respects the tensor product. So after this step one obtains a corresponding decomposition of \mathfrak{S} -modules. To get the decomposition of the filtered Dieudonné modules as claimed in the first isomorphism, one applies next Theorem 1.4.2 and Corollary 1.4.3 (i) loc. cit. to each factor in the previous decomposition of \mathfrak{S} -modules. Consider then the second isomorphism: By Corollary 3.7, one has a tensor decomposition of $\mathbb{Z}_{p^r}[\text{Gal}_E]$ -modules:

$$L_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigotimes_{i=0}^{r-1} L_{B, \sigma^i},$$

where $L_{B, \sigma^i} = L_B \otimes_{\mathbb{Z}_{p^r}, \sigma^i} \mathbb{Z}_{p^r}$, which is also equal to L_i in the eigen decomposition of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ with respect to $s_{B, et}$ in the proof of Lemma 4.3. Taking the r -th tensor power of

$$L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigoplus_{i=0}^{r-1} L_i,$$

and then the $\text{Gal}(\mathbb{Z}_{p^r}|\mathbb{Z}_p)$ -invariants of both sides, one gets a natural direct decomposition of $\mathbb{Z}_p[\text{Gal}_E]$ -modules:

$$L_B^{\otimes r} = L_{A_1} \oplus L'_{A_1}.$$

Thus L_{A_1} is naturally isomorphic to a Gal_E -sublattice of $L_B^{\otimes r}$. Proposition 3.16 shows that this sublattice is in fact of Hodge-Tate weights $\{0, 1\}$ with the induced filtered ϕ module structure given by Corollary 4.4. This implies the second isomorphism. \square

Put $G_A = G_{\mathbb{Z}_p}$ and $\mathcal{G}_A = G_A \times_{\mathbb{Z}_p} W(k) \subset \text{GL}_{W(k)}(M_A)$, the subgroup defined by the corresponding crystalline Tate cycles. Recall that after a conjugation by an element in $G(\mathbb{Q}_p)$, there is a central isogeny

$$\mathbb{Z}_p^* \times \prod_{i=1}^n \text{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}}) \rightarrow G_A.$$

Let G_{A_1} (resp. G_{A_2}) be the image of $\mathbb{Z}_p^* \times \text{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_1}})$ (resp. $\prod_{i=2}^n \text{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}})$) in $\text{GL}(L_{A_1})$ (resp. $\text{SL}(L_{A_2})$). By the construction of the tensor decomposition, one has the following commutative diagram:

$$\begin{array}{ccccc} \text{Gal}_E & \xrightarrow{\quad} & G_A & \xrightarrow{\quad \twoheadrightarrow \quad} & G_{A_1} \times G_{A_2} \\ & \searrow & \downarrow & \swarrow \quad \quad \quad & \downarrow \\ & & \text{GL}(L_{A_1}) \times \text{SL}(L_{A_2}) & \xrightarrow{\quad \otimes \quad} & \text{GL}(L_{A_1} \otimes L_{A_2}) \end{array}$$

Consider the group homomorphism

$$\otimes^r : \text{GL}(L_B) \rightarrow \text{GL}(L_B^{\otimes r}), g \mapsto (g^{\otimes r} : v_1 \otimes \cdots \otimes v_r \mapsto g(v_1) \otimes \cdots \otimes g(v_r)).$$

It is a central isogeny over the image.

Lemma 4.9. *The restriction of \otimes^r to the subgroup G_B factors*

$$\otimes^r|_{G_B} : G_B \rightarrow \text{GL}(L_{A_1}) \times \text{GL}(L'_{A_1}) \subset \text{GL}(L_B^{\otimes r}).$$

Proof. Recall that for a $g \in \text{GL}(L_B)$, $g \in G_B$ iff $g(s_B) = s_B$ up to a scalar. This implies that $g \otimes 1$ preserves the eigen decomposition of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$. So $\otimes^r(g \otimes 1)$ respects the direct sum decomposition

$$L_B^{\otimes r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = L_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} \oplus L'_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}.$$

Thus $\otimes^r(g)$ preserves the decomposition

$$L_B^{\otimes r} = L_{A_1} \oplus L'_{A_1}.$$

Hence the lemma follows. \square

Let $\xi_{et} : G_B \rightarrow \text{GL}(L_{A_1})$ be the composite of $\otimes^r|_{G_B}$ with the projection to the first factor in the above lemma. The reductive subgroup $G_{A_1} \subset \text{GL}(L_{A_1})$ is defined by a finite set of tensors in $L_{A_1}^{\otimes}$.

Lemma 4.10. *A tensor in $L_{A_1}^{\otimes n}$ is fixed by G_{A_1} only if $n = 2a$ is even, and it must be of form $\det(L_{A_1})^{\otimes a} \subset L_{A_1}^{\otimes n}$.*

Proof. Assume n positive. The G_{A_1} -action respects the tensor decomposition:

$$L_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \otimes_{i=0}^{r-1} L_i.$$

A tensor in $L_{A_1}^{\otimes n}$ is fixed by G_{A_1} is by definition a rank one \mathbb{Z}_p -subrepresentation of G_{A_1} . So it gives rise to a rank one \mathbb{Z}_{p^r} -subrepresentation of G_{A_1} in $[\otimes_{i=0}^{r-1} L_i]^{\otimes n}$. Embed $\mathbb{Z}_{p^r} \hookrightarrow \mathbb{C}_p$, and choose any lifting of σ in $\text{Gal}_{\mathbb{Q}_p}$ and then extend it continuously to an automorphism of \mathbb{C}_p . Call it again σ . A rank one \mathbb{Z}_{p^r} -subrepresentation gives a one dimensional subrepresentation of $\text{GL}_2(\mathbb{C}_p)$ in $[\otimes_{i=0}^{r-1} L_i \otimes \mathbb{C}_p]^{\otimes n}$. Recall that the G_{A_1} -action on L_i is isomorphic to the i -th σ -conjugate of the standard action of $\text{GL}_2(\mathbb{Z}_{p^r})$ on $\mathbb{Z}_{p^r}^{\oplus 2}$. Then we study the $\text{GL}_2(\mathbb{C}_p)$ -invariant lines in $[\otimes_{i=0}^{r-1} (\mathbb{C}_p^{\oplus 2})_i]^{\otimes n}$, where $(\mathbb{C}_p^{\oplus 2})_i := \mathbb{C}_p^{\oplus 2} \otimes_{\mathbb{C}_p, \sigma^i} \mathbb{C}_p$. For that we apply the standard finite dimensional representation theory of complex Lie groups (see [14] Ch 6). For a partition λ of n , one has the irreducible decomposition of $\prod_{i=0}^{r-1} \text{GL}_2(\mathbb{C}_p)$ -modules:

$$\mathbb{S}_\lambda[\otimes_{i=0}^{r-1} (\mathbb{C}_p^{\oplus 2})_i] = \bigoplus_{\lambda_0, \dots, \lambda_{r-1}} C_{\lambda_0 \dots \lambda_{r-1} \lambda} \cdot \mathbb{S}_{\lambda_0}(\mathbb{C}_p^{\oplus 2})_0 \otimes \dots \otimes \mathbb{S}_{\lambda_{r-1}}(\mathbb{C}_p^{\oplus 2})_{r-1},$$

where λ_i in the summation runs through all possible partitions of n . As $\dim(\mathbb{C}_p^{\oplus 2})_i = 2$, the only possible λ_i s are of form $\{n - a, a\}$ for $a \leq \frac{n}{2}$, and

$$\mathbb{S}_{\{n-a, a\}}(\mathbb{C}_p^{\oplus 2})_i = \begin{cases} \mathbb{S}_{\{n-2a\}}(\mathbb{C}_p^{\oplus 2})_i = \text{Sym}^{n-2a}(\mathbb{C}_p^{\oplus 2})_i \otimes [\det(\mathbb{C}_p^{\oplus 2})_i]^a, & \text{if } 2a < n; \\ \mathbb{S}_{\{a, a\}}(\mathbb{C}_p^{\oplus 2})_i = [\det(\mathbb{C}_p^{\oplus 2})_i]^a, & \text{if } 2a = n. \end{cases}$$

Since $\text{GL}_2(\mathbb{C}_p)$ is embedded into $\prod_{i=0}^{r-1} \text{GL}_2(\mathbb{C}_p)$ via $g \mapsto (g, \sigma g, \dots, \sigma^{r-1} g)$, the above decomposition is also irreducible with respect to $\text{GL}_2(\mathbb{C}_p)$ -action. Summarizing these discussions, we conclude that there exists a G_{A_1} -invariant tensor s_α in $L_{A_1}^{\otimes n}$ only if $n = 2a$ is even, and $s_\alpha \otimes 1 \in [\otimes_{i=0}^{r-1} L_i]^{\otimes n}$ is of form $\otimes_{i=0}^{r-1} [\det(L_i)]^a$, which implies that $s_\alpha \in \det(L_{A_1})^{\otimes a}$. \square

Corollary 4.11. *The morphism ξ_{et} factors*

$$\xi_{et} : G_B \rightarrow G_{A_1} \subset \text{GL}(L_{A_1}),$$

and the induced morphism $\xi_{et} : G_B \rightarrow G_{A_1}$ is a central isogeny.

Proof. For n an even natural number. Let $s_\alpha \in L_{A_1}^{\otimes n}$ be a tensor for G_{A_1} . It is to show that the image of G_B under ξ_{et} fixes s_α . By Lemma 4.10,

$$s_\alpha \otimes 1 = \otimes_{i=0}^{r-1} [\det(L_i)]^{\frac{n}{2}}.$$

It is clear that for a $g \in G_B$, $\otimes^r(g) \otimes 1$ stabilizes the line $\otimes_{i=0}^{r-1} [\det(L_i)]^{\frac{n}{2}}$. This implies that $\otimes^r(g)$ stabilizes s_α . As ξ_{et} is a central isogeny over its image and both G_B and G_{A_1} are isomorphic to $\text{GL}_2(\mathbb{Z}_{p^r})$, ξ_{et} induces a central isogeny from G_B to G_{A_1} . \square

So we have the following central isogenies of groups over \mathbb{Z}_p :

$$G_B \times G_{A_2} \twoheadrightarrow G_{A_1} \times G_{A_2} \leftarrow G_A.$$

Put $\mathcal{G}_{A_i} = G_{A_i} \times_{\mathbb{Z}_p} W(k)$. Taking the base change to $W(k)$ one obtains central isogenies of groups over $W(k)$:

$$\mathcal{G}_B \times \mathcal{G}_{A_2} \xrightarrow{\xi_1} \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \xleftarrow{\xi_2} \mathcal{G}_A.$$

For a certain natural number l , the cocharacter $\mathbb{G}_m \rightarrow \mathcal{G}_{A_1} \times \mathcal{G}_{A_2}$, which is the composite

$$\mathbb{G}_m \xrightarrow{x \mapsto x^l} \mathbb{G}_m \xrightarrow{\mu_{B'} \times \text{id}} \mathcal{G}_B \times \mathcal{G}_{A_2} \xrightarrow{\xi_1} \mathcal{G}_{A_1} \times \mathcal{G}_{A_2},$$

lifts to a cocharacter $\nu : \mathbb{G}_m \rightarrow \mathcal{G}_A$. By Proposition 4.8, the reduction of ν modulo p induces the same filtration as given by $\text{Fil}_A^1 \otimes k$ on $M_A \otimes k$. Then the filtration on M_A defined by ν corresponds to a p -divisible group A' over $W(k)$ lifting the p -divisible $A \otimes k$ over k . We call ν $\mu_{A'}$ in the following. One discusses the cocharacter $\xi_1 \circ \mu_{B'}$ similarly and obtains then a p -divisible group A'_1 over $W(k)$ lifting $A_1 \otimes k$. It follows that one has an isomorphism of filtered ϕ modules similar to that in Proposition 4.8 for the filtered Dieudonné module of A by replacing A_1 with A'_1 and B in Corollary 4.4 with B' .

Consider the opposite unipotents $U_{\mathcal{G}_B} \times \text{id}$ (resp. $U_{\mathcal{G}_{A_1}} \times \text{id}$ and $U_{\mathcal{G}_A}$) induced by the cocharacter $\mu_{B'} \times \text{id}$ (resp. $\xi_1 \circ (\mu_{B'} \times \text{id})$ and $\mu_{A'}$). By the construction, ξ_1 (resp. ξ_2) restricts to an isogeny from $U_{\mathcal{G}_B} \times \text{id}$ to $U_{\mathcal{G}_{A_1}} \times \text{id}$. (resp. from $U_{\mathcal{G}_A}$ to $U_{\mathcal{G}_{A_1}} \times \text{id}$). Thus taking the completion along the identity section, one obtains an isomorphism

$$\hat{\xi}_{\text{cris}} = \hat{\xi}_2^{-1} \circ \hat{\xi}_1 : \hat{U}_{\mathcal{G}_B} \xrightarrow{\sim} \hat{U}_{\mathcal{G}_A}.$$

Let $(\mathcal{N}_A, \text{Fil}_{\mathcal{N}_A}^1, \nabla_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ be the following filtered Dieudonné module over $\hat{U}_{\mathcal{G}_A}$: Let $R_{\mathcal{G}_A}$ be the complete local ring of $\hat{U}_{\mathcal{G}_A}$ and $\phi_{\hat{U}_{\mathcal{G}_A}} : R_{\mathcal{G}_A} \rightarrow R_{\mathcal{G}_A}$ be the lifting of the absolute Frobenius obtained by pulling back the $\phi_{\hat{U}_{\mathcal{G}_B}}$ via $\hat{\xi}_{\text{cris}}^{-1}$. The triple

$$(\mathcal{N}_A = M'_A \otimes_{W(k)} R_{\mathcal{G}_A}, \text{Fil}_{\mathcal{N}_A}^1 = \text{Fil}_{M'_A}^1 \otimes_{W(k)} R_{\mathcal{G}_A}, \phi_{\mathcal{N}_A} = u \circ (\phi_{M'_A} \otimes \phi_{\hat{U}_{\mathcal{G}_A}})),$$

where u is the tautological $R_{\mathcal{G}_A}$ -point of $U_{\mathcal{G}_A}$, together with the connection $\nabla_{\mathcal{N}_A}$ deduced from Theorem 10 [13], makes the four tuple $(\mathcal{N}_A, \text{Fil}_{\mathcal{N}_A}^1, \nabla_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ an object in $MF_{[0,1]}^\nabla(R_{\mathcal{G}_A})$. We denote again by $\hat{\xi}_{\text{cris}}$ the equivalence of categories from $MF_{[0,1]}^\nabla(\hat{U}_{\mathcal{G}_B})$ to $MF_{[0,1]}^\nabla(\hat{U}_{\mathcal{G}_A})$ induced by the isomorphism $\hat{\xi}_{\text{cris}}$.

Theorem 4.12. *One has a natural isomorphism in the category $MF_{[0,1]}^\nabla(\hat{M}_{x_0})$:*

$$(H, F, \phi, \nabla)|_{\hat{M}_{x_0}} \cong \{\hat{\xi}_{\text{cris}}[\otimes_{i=0}^{r-1}(\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \nabla_{\mathcal{N}_i}), \phi_{\text{ten}}] \otimes (M_{A_2}, \text{Fil}_{A_2}^1, \phi_{A_2}, d)\}^{\oplus 2^{\epsilon(D)}},$$

where $[\otimes_{i=0}^{r-1}(\mathcal{N}_i, \text{Fil}_{\mathcal{N}_i}^1, \nabla_{\mathcal{N}_i}), \phi_{\text{ten}}] \in MF_{[0,1]}^\nabla(\hat{U}_{\mathcal{G}_B})$ is the one introduced after Proposition 4.7 and $(M_{A_2}, \text{Fil}_{A_2}^1, \phi_{A_2}, d)$ is a constant unit crystal with the trivial connection.

Proof. From Proposition 2.3.5 [18] and its proof, one knows that $\hat{M}_{x_0} = \hat{U}_{\mathcal{G}_A}$ is the deformation space of the p -divisible group A_0 with Tate cycles $\subset M_A^{\otimes}$ fixed by the group $\mathcal{G}_A \subset \text{GL}_{W(k)}(M_A)$. By the remarks of Faltings §7 [13], the above four tuple $(\mathcal{N}_A, \text{Fil}_{\mathcal{N}_A}^1, \nabla_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ gives an explicit description in the category $MF_{[0,1]}^\nabla(\hat{M}_{x_0})$ of the restriction $(H, F, \phi, \nabla)|_{\hat{M}_{x_0}}$. The decomposition of the triple $(\mathcal{N}_A, \text{Fil}_{\mathcal{N}_A}^1, \phi_{\mathcal{N}_A})$ follows from the above description of the universal filtered Dieudonné module and the corresponding statement of Proposition 4.8.

Then the connection decomposes accordingly: We equip the decomposition with the connection

$$\nabla_{dec} := (\otimes_{i=0}^{r-1} \nabla_{\mathcal{N}_i} \otimes d)^{\oplus 2^{\epsilon(D)}}.$$

Then the decomposition of $\phi_{\mathcal{N}_A}$ shows that it is horizontal with respect to both ∇_{dec} and $\nabla_{\mathcal{N}_A}$. By the uniqueness of such a connection (see proof of Theorem 10 [13], 4.3.2 [22]), $\nabla_{\mathcal{N}_A}$ is isomorphic to ∇_{dec} as claimed. \square

We state the following consequence for a later use:

Corollary 4.13. *One has an isomorphism in the category $MF_{big,r}^{\nabla}(\hat{M}_{x_0})$:*

$$(H, F, \phi^r, \nabla)|_{\hat{M}_{x_0}} \cong \{\hat{\xi}_{cris} \otimes_{i=0}^{r-1} (\mathcal{N}_i, Fil_{\mathcal{N}_i}^1, \phi_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}) \otimes (M_{A_2}, Fil_{A_2}^1, \phi_{A_2}^r, d)\}^{\oplus 2^{\epsilon(D)}}.$$

Remark 4.14. We expect a tensor decomposition of the universal filtered Dieudonné module attached to f over the global base M , in a form analogous to Theorem 4.12 and Corollary 4.13. In particular, there is a direct-tensor decomposition of (H, F, ϕ^r, ∇) in the category $MF_{big,r}^{\nabla}(M)$ (see §5.1) in the above form, which has an implication on the existence of Hasse-Witt pair. The problem is closely related to that of Remark 5.4. Since we are not able to show it under the current circumstance, we have to consider a certain direct summand contained in its second symmetric/wedge power, from which some technicalities arise, in order to construct a *global* object, the so-called weak Hasse-Witt pair in the next section.

5. A WEAK HASSE-WITT PAIR AND A MASS FORMULA

Let $f_0 : X_0 \rightarrow M_0$ be the reduction of the universal abelian scheme modulo \mathfrak{p} . In this section we construct a certain line bundle \mathcal{P}_0 of negative degree over M_0 together with a morphism $\tilde{F}_{rel} : F_{M_0}^{*r} \mathcal{P}_0 \rightarrow \mathcal{P}_0$ which is closely related with the relative Frobenius of f_0 . The pair $(\mathcal{P}_0, \tilde{F}_{rel})$ will be called a *weak Hasse-Witt pair*: The name is suggested by the classical Hasse-Witt map in the case of elliptic curves over char p . It is well known that for a modular family of elliptic curves $g : E \rightarrow S$ in char p , the Hasse-Witt map is a morphism

$$F_{rel} : F_S^* R^1 g_* \mathcal{O}_E \rightarrow R^1 g_* \mathcal{O}_E$$

induced by the relative Frobenius of g . The Hasse invariant is the section of the line bundle $R^1 g_* \mathcal{O}_E \otimes (F_S^* R^1 g_* \mathcal{O}_E)^{-1}$ given by F_{rel} . A classical theorem of Deuring and Igusa asserts that the zeroes of the Hasse invariant are of multiplicity one, and hence its zero divisor is the supersingular locus of S . One can use this to deduce further the classical Eichler-Deuring mass formula on the number of supersingular j -invariants by considering a semi-stable model of the Legendre family of elliptic curves. For the current family f_0 of abelian varieties, one has also the Hasse-Witt map

$$F_{rel} : F_{M_0}^* R^1 f_{0*} \mathcal{O}_{X_0} \rightarrow R^1 f_{0*} \mathcal{O}_{X_0}.$$

However the relation between the degeneracy of F_{rel} and the supersingular locus is far from obvious. As a remark, the determinant of F_{rel} is zero if there is no ordinary abelian variety as a closed fiber, which is the case for $r \geq 2$. A crucial step in our approach to a mass formula is the construction of a *weak* Hasse-Witt

pair $(\mathcal{P}_0, \tilde{F}_{rel})$ to the effect that the zero divisor of the section of $\mathcal{P}_0 \otimes F_{M_0}^{*r} \mathcal{P}_0^{-1}$ defined by \tilde{F}_{rel} coincides with the supersingular locus *up to multiplicity*.

5.1. Preliminary discussion. Everything discussed in this paragraph, if no mistake arises, is entirely due to Faltings ([12],[13]). We collect his results in a form which we could apply in the following conveniently. Note also that M in the following discussion could be relaxed to be an arbitrary smooth proper scheme over $W(k)$. Let $U = \text{Spec} R \subset M$ be a small affine subset, which means that there is an étale map $W(k)[T^\pm] \rightarrow R$. Let \bar{R} be the maximal extension of R which is étale in characteristic zero (see Ch II a) [12]) and $\Gamma_R = \text{Gal}(\bar{R}|R)$ be the Galois group. Let $MF_{[0,p-2]}^\nabla(R)$ be the category introduced in §3 [13], and $\text{Rep}_{\mathbb{Z}_p}(\Gamma_R)$ the category of continuous representations of Γ_R on free \mathbb{Z}_p -modules of finite rank. By the fundamental theorem (Theorem 5* [13]), there is a fully faithful contravariant functor

$$\mathbf{D} : MF_{[0,p-2]}^\nabla(R) \rightarrow \text{Rep}_{\mathbb{Z}_p}(\Gamma_R).$$

An object lying in the image of the functor \mathbf{D} is called a *dual crystalline representation*⁴. For our convenience, we shall also consider the covariant functor \mathbf{D}^* , which maps an object $H \in MF_{[0,p-2]}^\nabla(R)$ to the dual of $\mathbf{D}(H)$ in $\text{Rep}_{\mathbb{Z}_p}(\Gamma_R)$, and call an object in the image of \mathbf{D}^* a crystalline representation. The p -torsion analogue of the above theorem is established in [12]. For clarity of exposition, we use the subscript tor to distinguish the torsion analogues. So there is also a fully faithful functor (Theorem 2.6 in loc. cit.)

$$\mathbf{D}_{tor} : MF_{[0,p-2]}^\nabla(R)_{tor} \rightarrow \text{Rep}_{\mathbb{Z}_p}(\Gamma_R)_{tor}.$$

It follows from the construction that for an object $H \in MF_{[0,p-2]}^\nabla(R)$, one has $\mathbf{D}(H) = \lim_{\leftarrow n} \mathbf{D}_{tor}(\frac{H}{p^n H})$. Faltings has defined an adjoint functor \mathbf{E}_{tor} of \mathbf{D}_{tor} (see Ch II. f)-g) [12]). For an object $\mathbb{L} \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_R)$, one defines

$$\mathbf{E}(\mathbb{L}) := [\lim_{\leftarrow n} \mathbf{E}_{tor}(\frac{\mathbb{L}}{p^n \mathbb{L}})] / \text{torsion}.$$

Clearly, for $\mathbb{L} = \mathbf{D}(H)$, it holds that

$$\mathbf{E}(\mathbb{L}) = \lim_{\leftarrow n} \mathbf{E}_{tor}(\frac{\mathbb{L}}{p^n \mathbb{L}}) = \lim_{\leftarrow n} \frac{H}{p^n H} = H.$$

Finally define $\mathbf{E}^*(\mathbb{L}) := \mathbf{E}(\mathbb{L}^*)$.

Lemma 5.1. *Suppose $\mathbb{W}, \mathbb{W}_1, \mathbb{W}_2 \in \text{Rep}_{\mathbb{Z}_p}(\Gamma_R)$. The following basic properties hold:*

- (i) *Suppose $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$. Then \mathbb{W} is crystalline if and only if each \mathbb{W}_i is so.*
- (ii) *Suppose \mathbb{W} crystalline and a Schur functor \mathbb{S}_λ with λ a partition of $n \leq p-1$. Then $\mathbb{S}_\lambda(\mathbb{W})$ is still crystalline, and there is a natural isomorphism $\mathbf{E}^*(\mathbb{S}_\lambda \mathbb{W}) \simeq \mathbb{S}_\lambda \mathbf{E}^*(\mathbb{W})$.*

⁴It is said to be dual because the functor \mathbf{D} maps the first crystalline cohomology of an abelian variety to the dual of the first étale cohomology. See Theorem 7 [13].

- (iii) Suppose $\mathbb{W}_i, i = 1, 2$ crystalline. Then $\mathbb{W}_1 \otimes \mathbb{W}_2$ is crystalline, and there is a natural isomorphism

$$\mathbf{E}^*(\mathbb{W}_1 \otimes \mathbb{W}_2) \cong \mathbf{E}^*(\mathbb{W}_1) \otimes \mathbf{E}^*(\mathbb{W}_2).$$

Proof. Consider $\mathbf{E}_{\text{tor}}(\mathbb{W}/p^n) = \mathbf{E}_{\text{tor}}(\mathbb{W}_1/p^n) \oplus \mathbf{E}_{\text{tor}}(\mathbb{W}_2/p^n)$. By Ch II g) [12], it holds always

$$l(\mathbf{E}_{\text{tor}}(\mathbb{W}_i/p^n)) \leq l(\mathbb{W}_i/p^n),$$

and the equality holds iff \mathbb{W}_i/p^n lies in the image of \mathbf{D}_{tor} . Now assume \mathbb{W} to be dual crystalline. That is $\mathbb{W} = \mathbf{D}(H)$. So $\frac{\mathbb{W}}{p^n \mathbb{W}} = \frac{\mathbf{D}(H)}{\mathbf{D}(p^n H)} = \mathbf{D}_{\text{tor}}(\frac{H}{p^n H})$. Hence from

$$l(\mathbb{W}/p^n) = \sum_i l(\mathbb{W}_i/p^n) \geq \sum_i l(\mathbf{E}_{\text{tor}}(\mathbb{W}_i/p^n)) = l(\mathbf{E}_{\text{tor}}(\mathbb{W}/p^n)),$$

it follows that there are $H_{i,n} \in MF_{[0,p-2]}^\nabla(R)_{\text{tor}}, i = 1, 2$ such that $\mathbf{D}_{\text{tor}}(H_{i,n}) = \mathbb{W}_i/p^n$ and by the faithfulness of \mathbf{D}_{tor} , $H_{1,n} \oplus H_{2,n} = H/p^n$. Taking the inverse limit, one obtains $H_i = \lim_{\infty \leftarrow n} H_{i,n}$ with the equality $H_1 \oplus H_2 = H$, which implies that H_i is torsion free and is an object in $MF_{[0,p-2]}^\nabla(R)$. Thus it follows that

$$\mathbf{D}(H_i) = \lim_{\infty \leftarrow n} \mathbf{D}_{\text{tor}}(H_i/p^n) = \lim_{\infty \leftarrow n} \mathbb{W}_i/p^n = \mathbb{W}_i,$$

and thereby \mathbb{W}_i is dual crystalline. The other direction of (i) is obvious. Clearly (ii) follows from (iii). To show (iii), it is to show that for $H_i \in MF_{[0,p-2]}^\nabla(R), i = 1, 2$, there is a natural isomorphism

$$\mathbf{D}(H_1) \otimes \mathbf{D}(H_2) \cong \mathbf{D}(H_1 \otimes H_2).$$

Taking an element $f_i \in \mathbf{D}(H_i)$, which is an \hat{R} -linear map from H_i to $B^+(R)$ respecting the filtrations and the ϕ s, one forms the \hat{R} -linear map $f_1 \otimes f_2 : H_1 \otimes H_2 \rightarrow B^+(R)$. It respects the filtrations and the ϕ s and therefore gives an element in $\mathbf{D}(H_1 \otimes H_2)$. So one has a natural map $\mathbf{D}(H_1) \otimes \mathbf{D}(H_2) \rightarrow \mathbf{D}(H_1 \otimes H_2)$, which is obviously injective. Because both sides have the same \mathbb{Z}_p -rank, it remains to show that the quotient $\mathbf{D}(H_1 \otimes H_2)/\mathbf{D}(H_1) \otimes \mathbf{D}(H_2)$ has no torsion. For that we pass to modulo p reduction and use the functor \mathbf{D}_{tor} . The same argument as above applied to H_i/p shows that the \mathbb{F}_p -linear map $\mathbf{D}_{\text{tor}}(H_1/p) \otimes \mathbf{D}_{\text{tor}}(H_2/p) \rightarrow \mathbf{D}_{\text{tor}}(H_1 \otimes H_2/p)$ is injective and therefore is bijective. This shows the non p -torsionness. \square

Let $\mathcal{U} = \{U\}$ be a small affine open covering of M . Theorem 2.3 [12] shows that the categories $\{MF_{[0,p-2]}^\nabla(U)\}_{U \in \mathcal{U}}$ glue into the category $MF_{[0,p-2]}^\nabla(M)$ by choosing a local Frobenius lifting ϕ_U for each U (the integrable connection ∇ provides then the transformations on the local sections over the overlaps). Furthermore Faltings explained that these various local functors \mathbf{D}_{tor} glue to a global one from $MF_{[0,p-2]}^\nabla(M)_{\text{tor}}$ to $\text{Rep}_{\mathbb{Z}_p}(\pi_1(M^0))_{\text{tor}}$ (see page 42 [12]). By passing to limit, one obtains a global functor $\mathbf{D} : MF_{[0,p-2]}^\nabla(M) \rightarrow \text{Rep}_{\mathbb{Z}_p}(\pi_1(M^0))$. An object in the image of \mathbf{D} is called a *dual crystalline sheaf*. Similarly, one defines \mathbf{D}^* and \mathbf{E}^* in the global setting and calls an object in the image of \mathbf{D}^* a crystalline sheaf. Now let \mathbb{W} be a crystalline sheaf of M^0 and H the corresponding filtered Frobenius crystal to \mathbb{W} (i.e. $\mathbf{D}^*(H) = \mathbb{W}$). Let x be a $W(k)$ -valued point of M . Consider

the specialization of both objects into the point x : Via the splitting of the short exact sequence

$$1 \rightarrow \pi_1(\bar{M}^0) \rightarrow \pi_1(M^0) \rightarrow \mathrm{Gal}_{\mathrm{Frac}(W(k))} \rightarrow 1$$

induced by the point $x^0 : \mathrm{Frac}(W(k)) \rightarrow M^0$, \mathbb{W}_{x^0} is a representation of $\mathrm{Gal}_{\mathrm{Frac}(W(k))}$. On the other hand, H_x is obviously an object in $MF_{[0,p-2]}(W(k))$.

Lemma 5.2. *Notation as above. Then the following statements hold:*

- (i) *The Galois representation $\mathbb{W}_{x^0} \otimes \mathbb{Q}_p$ is crystalline in the sense of Fontaine.*
- (ii) *H_x is naturally a strong divisible lattice of the $\mathrm{Frac}(W(k))$ -vector space $D_{\mathrm{crys}}(\mathbb{W}_{x^0} \otimes \mathbb{Q}_p)$ in the sense of Fontaine-Laffaille ([8]).*
- (iii) *There is a natural isomorphism of $\mathbb{Z}_p[\mathrm{Gal}_{\mathrm{Frac}(W(k))}]$ -modules:*

$$\mathbf{D}^*(H_x) \simeq \mathbf{D}^*(H)_{x^0}.$$

Consequently, there is a natural isomorphism in $MF_{[0,p-2]}(W(k))$:

$$\mathbf{E}^*(\mathbb{W})_x \cong \mathbf{E}^*(\mathbb{W}_{x^0}).$$

Proof. It is clear that we can pass the problem to a small affine subset $U = \mathrm{Spec} R \subset M$. Choose a local coordinate T of R such that the $W(k)$ -point x of R is given by $T = 1$ (i.e. the composite $W(k)[T^\pm] \rightarrow R \rightarrow W(k)$ is the $W(k)$ -morphism determined by $T \mapsto 1$). Fix an isomorphism $\bar{R} \otimes_R W(k) \cong \overline{W(k)}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spec} \overline{W(k)} & \xrightarrow{\bar{x}} & \mathrm{Spec} \bar{R} \\ \downarrow & & \downarrow \\ \mathrm{Spec} W(k) & \xrightarrow{x} & \mathrm{Spec} R. \end{array}$$

Note that the subgroup $\Gamma_{R,x} \subset \Gamma_R$ preserving the prime ideal $\ker(\bar{R} \rightarrow \overline{W(k)})$ of \bar{x} is naturally isomorphic to $\mathrm{Gal}_{\mathrm{Frac}(W(k))}$, and it is equal to the image of the splitting $\Gamma_R \twoheadrightarrow \mathrm{Gal}_{\mathrm{Frac}(W(k))}$ induced by the $W(k)$ -point x . Fix a Frobenius lifting ϕ of \hat{R} which fixes the point x (e.g the one determined by $T \mapsto T^p$). Note also that the point $\bar{x} : \bar{R} \rightarrow \overline{W(k)}$ induces a surjection of $B^+(W(k))$ -algebras $B^+(\hat{R}) \rightarrow B^+(W(k))$, which preserves the filtration and the Frobenius. Recall that

$$\mathbf{D}(H) = \mathrm{Hom}_{\hat{R}, \mathrm{Fil}, \phi}(H, B^+(\hat{R})) = (H^* \otimes_{\hat{R}} B^+(\hat{R}))^{\mathrm{Fil}=0, \phi=1},$$

and

$$\mathbf{D}(H_x) = \mathrm{Hom}_{W(k), \mathrm{Fil}, \phi}(H_x, B^+(W(k))) = (H_x^* \otimes_{W(k)} B^+(W(k)))^{\mathrm{Fil}=0, \phi=1}.$$

The above free \mathbb{Z}_p -modules (say of rank n) are basically obtained by solving certain equations (see page 127-128 [13], or page 37-38 [12]). There are also natural surjections

$$B^+(\hat{R}) \twoheadrightarrow B^+(\hat{R})/p \cdot B^+(\hat{R}) \twoheadrightarrow \bar{R}/p \cdot \bar{R},$$

and similarly for $B^+(W(k))$. These make the following diagrams commute:

$$\begin{array}{ccccc}
B^+(\hat{R}) & \longrightarrow & B^+(\hat{R})/p \cdot B^+(\hat{R}) & \longrightarrow & \bar{R}/p \cdot \bar{R} \\
\downarrow & & \downarrow & & \downarrow \\
B^+(W(k)) & \longrightarrow & B^+(W(k))/p \cdot B^+(W(k)) & \longrightarrow & \overline{W(k)}/p \cdot \overline{W(k)},
\end{array}$$

where the vertical arrows are induced by the point \bar{x} . Faltings showed in loc. cit. that it suffices to solve the equations over the quotient \bar{R}/p (resp. $\overline{W(k)}/p$) because each solution over the quotient can be uniquely lifted. Now choose a filtered basis $\{h_i\}$ of H , which restricts to a filtered basis of H_x . An element of $\mathbf{D}(H)$ is then given by an n -tuple in $B^+(\hat{R})$ satisfying a system of equations coming from the condition on ϕ s. For each such an n -tuple, we obtain an n -tuple in $B^+(W(k))$ by projecting each component to $B^+(W(k))$ (the projection $B^+(\hat{R}) \rightarrow B^+(W(k))$ induced by the point \bar{x}). As the projection preserves the filtration and the Frobenius, and as the filtration and the Frobenius on H_x are the one of H by restriction, any so-obtained n -tuple satisfies the equations required for $\mathbf{D}(H_x)$. So we have a \mathbb{Z}_p -linear map

$$ev_x : \mathbf{D}(H) \rightarrow \mathbf{D}(H_x), f \mapsto f(x).$$

Consider first the Galois action. Recall that $\text{Gal}_{\text{Frac}(W(k))}$ acts on $\mathbf{D}(H_x) \subset H_x^* \otimes_{W(k)} B^+(W(k))$ on the second tensor factor. But the Γ_R -action on $\mathbf{D}(H) \subset H^* \otimes_{\hat{R}} B^+(\hat{R})$ must be also intertwined with the connection ∇ on the first factor. However the restriction to the subgroup $\Gamma_{R,x}$ does not involve the connection (see Ch. II e) [12] for the p -torsion situation which we can also assume in the argument). So the above map ev_x is equivariant with respect to $\Gamma_{R,x}$ -action on $\mathbf{D}(H)$ and $\text{Gal}_{\text{Frac}(W(k))}$ action on $\mathbf{D}(H_x)$.

Next we claim that ev_x is a \mathbb{Z}_p -isomorphism. For that we consider the base change $ev_x \otimes \mathbb{Q}_p$ and then the reduction $ev_x \otimes \mathbb{F}_p$. By Ch II h) [12], one has a natural isomorphism

$$H \otimes_{\hat{R}} B(\hat{R}) \cong \mathbf{D}^*(H) \otimes_{\mathbb{Z}_p} B(\hat{R}),$$

which respects the Γ_R -actions, filtrations and ϕ s. Tensoring the above isomorphism with $B(W(k))$ as $B(\hat{R})$ -modules (the morphism $B(\hat{R}) \rightarrow B(W(k))$ induced by \bar{x}) and taking the $\Gamma_{R,x}$ -invariance of both sides, we obtain an isomorphism of $\text{Gal}_{\text{Frac}(W(k))}$ -representations:

$$V_{\text{crys}}(H_x \otimes \text{Frac}(W(k))) \cong \mathbf{D}^*(H)_{x^0} \otimes \mathbb{Q}_p.$$

That is, there is a natural isomorphism

$$\mathbf{D}(H)_{x^0} \otimes \mathbb{Q}_p \cong V_{\text{crys}}^*(H_x \otimes \text{Frac}(W(k))).$$

By Fontaine-Laffaille (see §7-8 in [8], see also §2 [4]), $\mathbf{D}(H_x)$ is a Galois lattice of $V_{\text{crys}}^*(H_x \otimes \text{Frac}(W(k)))$ by the isomorphism, and H_x is a strong divisible lattice of $D_{\text{crys}}(\mathbf{D}^*(H)_{x^0} \otimes \mathbb{Q}_p)$. This shows (i) and (ii). Also it implies that the map $ev_x \otimes \mathbb{Q}_p$ is an isomorphism. In particular the map ev_x is injective. Using the fact that the composite $B^+(W(k)) \rightarrow B^+(\hat{R}) \rightarrow B^+(W(k))$ is

the identity, one sees that $ev_x(\mathbf{D}(H)) \cap p\mathbf{D}(H_x) = ev_x(p\mathbf{D}(H))$, and the map $ev_x \otimes \mathbb{F}_p : \mathbf{D}(H)/p\mathbf{D}(H) \rightarrow \mathbf{D}(H_x)/p\mathbf{D}(H_x)$ is therefore injective. Now that the \mathbb{F}_p -vector spaces $\mathbf{D}(H)/p\mathbf{D}(H)$ and $\mathbf{D}(H_x)/p\mathbf{D}(H_x)$ have the same dimension n , $ev_x \otimes \mathbb{F}_p$ is an isomorphism. Thus ev_x is an isomorphism. This proves (iii). \square

Let $r \in \mathbb{N}$ be a natural number. Let $\text{Rep}_{\mathbb{Z}_{p^r}}(\pi_1(M^0)) \subset \text{Rep}_{\mathbb{Z}_p}(\pi_1(M^0))$ be the full subcategory of $\mathbb{Z}_{p^r}[\pi_1(M^0)]$ -modules. An object which lies in both $\text{Rep}_{\mathbb{Z}_{p^r}}(\pi_1(M^0))$ and the image of \mathbf{D}^* is called a \mathbb{Z}_{p^r} -crystalline sheaf. One notes that the proof of Theorem 2.3 [12] works verbatim to show that the local categories $\{MF_{big,r}^\nabla(U)\}_{U \in \mathcal{U}}$ (see §4.1) glue into a global category $MF_{big,r}^\nabla(M)$. A typical object in this category is obtained by replacing the Frobenius of an object in $MF_{[0,p-2]}^\nabla(M)$ with its r -th power.

Lemma 5.3. *Let \mathbb{W} be a \mathbb{Z}_{p^r} -crystalline sheaf. Assume that $\mathbb{Z}_{p^r} \subset \mathcal{O}_M$. Then there is a natural decomposition in the category $MF_{big,r}^\nabla(M)$:*

$$\mathbf{E}^*(\mathbb{W}) = \bigoplus_{i=0}^{r-1} \mathbf{E}^*(\mathbb{W})_i.$$

Proof. The multiplication by $s \in \mathbb{Z}_{p^r}$ on \mathbb{W} commutes with $\pi_1(M^0)$ -action. Hence it gives rise to an endomorphism s_{MF} of $\mathbf{E}^*(\mathbb{W})$ in the category $MF_{[0,p-2]}^\nabla(M)$. By assumption \mathcal{O}_M contains the eigenvalues of s_{MF} . The eigen decomposition of $\mathbf{E}^*(\mathbb{W})$ with respect to s_{MF} gives rise to a decomposition of form $\bigoplus_{i=0}^{r-1} \mathbf{E}^*(\mathbb{W})_i$ such that the direct factors are preserved by ∇ and permutes cyclically by ϕ . Hence the lemma follows. \square

5.2. Second wedge/symmetric power of the universal filtered Dieudonné module. From now on the rational prime number p is assumed to be ≥ 5 in addition to Assumption 2.2. The aim of the paragraph is to show a direct sum decomposition of the second wedge (resp. symmetric) of the universal filtered Dieudonné crystal for d an even (resp. odd) number. Recall from Corollary 3.6 we have a tensor decomposition of étale local systems

$$\mathbb{H} = (\mathbb{V} \otimes \mathbb{U})^{\oplus 2^{\epsilon(D)}}.$$

It follows from Lemma 5.1 that the direct summand $\mathbb{H}' := \mathbb{V} \otimes \mathbb{U}$ of \mathbb{H} is crystalline. As $\det(\mathbb{H}) \cong \mathbb{Z}_p(-2^{d+\epsilon(D)-1})$, it follows that

$$\det \mathbb{H}' \simeq \mathbb{Z}_p(-2^{d-1}).$$

Consider the following \mathbb{Z}_{p^d} -crystalline sheaf

$$\tilde{\mathbb{H}}' := \mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^d}(2^{d-2}) = (\mathbb{V} \otimes \det(\mathbb{V})^{-\frac{1}{2}} \otimes \mathbb{U} \otimes \det(\mathbb{U})^{-\frac{1}{2}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^d}.$$

For $1 \leq i \leq r$, put

$$\tilde{\mathbb{V}}_i = \mathbb{V}_{1,\sigma^{i-1}} \otimes \det(\mathbb{V}_1)^{-\frac{1}{2}}, \quad \tilde{\mathbb{V}}'_i = \tilde{\mathbb{V}}_i \otimes_{\mathbb{Z}_{p^r}} \mathbb{Z}_{p^d},$$

and for $r+1 = r_1+1 \leq i \leq r_1+r_2$, put

$$\tilde{\mathbb{V}}'_i = (\mathbb{U}_{1,\sigma^{i-1}} \otimes \det(\mathbb{U}_1)^{-\frac{1}{2}}) \otimes_{\mathbb{Z}_{p^{r_2}}} \mathbb{Z}_{p^d},$$

and so on. By Corollary 3.6, we have a decomposition of $\tilde{\mathbb{H}}'$ into tensor product of rank two \mathbb{Z}_{p^d} -étale local systems:

$$\tilde{\mathbb{H}}' = \otimes_{i=1}^d \tilde{\mathbb{V}}'_i.$$

In the tensor decomposition, we assume that the factor $\tilde{\mathbb{V}}_1$ corresponds to the place τ (see Lemma 3.1).

Remark 5.4. We expect that each tensor factor $\tilde{\mathbb{V}}'_i$ in the above decomposition is a \mathbb{Z}_{p^d} -crystalline sheaf. The next lemma shows that $\text{Sym}^2 \tilde{\mathbb{V}}'_i$ is a direct factor of a crystalline sheaf and therefore crystalline by Lemma 5.1 (i).

The following lemma is proved by induction on d :

Lemma 5.5. *For $I = (i_1, \dots, i_l)$ a multi-index in $\{1, \dots, d\}$, put*

$$\text{Sym}^2(\tilde{\mathbb{V}}')_I := \otimes_{j=1}^l \text{Sym}^2 \tilde{\mathbb{V}}'_{i_j}.$$

One has a direct sum decomposition of \mathbb{Z}_{p^d} -étale local systems:

(i) *for d even,*

$$\bigwedge^2(\tilde{\mathbb{H}}') = \bigoplus_{I, |I| \text{ odd}} \text{Sym}^2(\tilde{\mathbb{V}}')_I, \quad \text{Sym}^2(\tilde{\mathbb{H}}') = \bigoplus_{I, |I| \text{ even}} \text{Sym}^2(\tilde{\mathbb{V}}')_I,$$

(ii) *for d odd,*

$$\bigwedge^2(\tilde{\mathbb{H}}') = \bigoplus_{I, |I| \text{ even}} \text{Sym}^2(\tilde{\mathbb{V}}')_I, \quad \text{Sym}^2(\tilde{\mathbb{H}}') = \bigoplus_{I, |I| \text{ odd}} \text{Sym}^2(\tilde{\mathbb{V}}')_I.$$

In the following we shall focus on the direct summand $\text{Sym}^2(\tilde{\mathbb{V}}_1)$ in the decomposition since it is so to speak the (rank three) uniformizing direct factor of the weight two integral p -adic variation of Hodge structures of the universal family. Also one notices that this factor is actually defined over \mathbb{Z}_{p^r} ⁵. So by taking the $\text{Gal}(\mathbb{Z}_{p^d}|\mathbb{Z}_{p^r})$ -invariants, one obtains a direct decomposition into \mathbb{Z}_{p^r} -dual crystalline sheaves with $\text{Sym}^2 \tilde{\mathbb{V}}_1$ as a direct factor for the second wedge (resp. symmetric) power for n even (resp. odd).

Proposition 5.6. *Let $(H', F, \phi, \nabla) \in MF_{[0,1]}^\nabla(M)$ be the sub filtered F -crystal corresponding to the factor $\mathbb{H}' \subseteq \mathbb{H}$. One has a direct sum decomposition in $MF_{big,r}^\nabla(M)$:*

(i) *for d even,*

$$\bigwedge^2(H', F, \phi^r, \nabla) = \bigoplus_{i=0}^{r-1} \mathbf{E}^*(\text{Sym}^2 \tilde{\mathbb{V}}_i)_0 \{-2^{d-1}\} \oplus \text{rest term}.$$

⁵It is important to work with \mathbb{Z}_{p^r} instead of \mathbb{Z}_{p^d} -coefficients because we do not have the inclusion $\mathbb{Z}_{p^d} \subset \mathcal{O}_M$ in general. See Lemma 5.3.

(ii) for d odd,

$$\mathrm{Sym}^2(H', F, \phi^r, \nabla) = \bigoplus_{i=0}^{r-1} \mathbf{E}^*(\mathrm{Sym}^2 \tilde{\mathbb{V}}_i)_0 \{-2^{d-1}\} \oplus \text{rest term}.$$

Proof. We shall prove (i) only ((ii) is similar). By the discussion before the proposition, we obtain a decomposition in $MF_{[0,2]}^\nabla(M)$:

$$\mathbf{E}^*\left(\bigwedge^2(\mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}(2^{d-2}))\right) = \bigoplus_{i=0}^{r-1} \mathbf{E}^*(\mathrm{Sym}^2 \tilde{\mathbb{V}}_i) \oplus \text{rest term}.$$

By Lemma 5.1 (ii), the left hand side is equal to $\bigwedge^2(H'\{2^{d-2}\})$. The claimed decomposition is obtained by considering the eigen decomposition of both sides corresponding the eigenvalue s_0 : The argument is similar to that of Lemma 4.3. The right hand side is clear, and the question is the left hand side. It suffices to consider the eigen component after inverting p . By Ch II h) [12], one has a Γ_R -isomorphism

$$\mathbf{E}^*\left(\bigwedge^2(\mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}|_{\hat{U}}) \otimes_{\hat{R}} B(R)\right) \cong \bigwedge^2(\mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}|_{\hat{U}}) \otimes_{\mathbb{Z}_p} B(R).$$

It follows that

$$\begin{aligned} [\mathbf{E}^*\left(\bigwedge^2(\mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})\right)|_{\hat{U}}]_0\left[\frac{1}{p}\right] &\cong [\bigwedge^2(\mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}|_{\hat{U}}) \otimes_{\mathbb{Z}_{p^r}} B(R)]^{\Gamma_R} \\ &\cong [\bigwedge^2 \mathbb{H}' \otimes_{\mathbb{Z}_p} B(R)]^{\Gamma_R} \\ &\cong \mathbf{E}^*\left(\bigwedge^2 \mathbb{H}'\right)\left[\frac{1}{p}\right] \\ &\cong \bigwedge^2(\mathbf{E}^*(\mathbb{H}'))\left[\frac{1}{p}\right] \end{aligned}$$

This shows that the eigen submodule of $\mathbf{E}^*(\bigwedge^2(\mathbb{H}' \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}))$ to the eigenvalue s_0 is naturally isomorphic to $\bigwedge^2 H'$ as claimed. \square

5.3. Construction of a weak Hasse-Witt pair. In the following \mathbf{E}_0 denotes the factor $\mathbf{E}^*(\mathrm{Sym}^2 \tilde{\mathbb{V}}_1)_0 \{-2^{d-1}\}$ in the decomposition of Proposition 5.6. Let $x_0 \in M_0(k)$ be a k -rational point and x a $W(k)$ -valued point of M lifting x_0 .

Lemma 5.7. *One has a natural isomorphism in the category $MF_{big,r}^\nabla(\hat{M}_{x_0})$:*

$$\mathbf{E}_0|_{\hat{M}_{x_0}} \cong \hat{\xi}_{crys}[\mathrm{Sym}^2 \mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)].$$

Proof. Assume d to be even. By Corollary 4.13, one has a natural isomorphism in $MF_{big,r}^\nabla(\hat{M}_{x_0})$:

$$H'|_{\hat{M}_{x_0}} \cong \hat{\xi}_{crys}(\bigotimes_{i=0}^{r-1} \mathcal{N}_i) \otimes M_{A_2}.$$

By a Schur functor calculation as in Lemma 5.5, one finds that via the isomorphism $\hat{\xi}_{crys}[\mathrm{Sym}^2 \mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)]$ is a direct factor of $\bigwedge^2(H'|_{\hat{M}_{x_0}})$. The point is to show that it is the direct factor $\mathbf{E}_0|_{\hat{M}_{x_0}}$. Note that $\hat{\xi}_{crys}[\mathrm{Sym}^2 \mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)]$ is the unique rank three direct factor with the nontrivial filtration. So it suffices to show that the rank three direct factor $\mathbf{E}_0|_{\hat{M}_{x_0}}$ has also this property. To that we show that the filtration of the filtered ϕ^r -module $(\mathbf{E}_0)_x \otimes \mathrm{Frac}W(k)$ is nontrivial. By Lemma 5.5 $\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2}))$ is a direct factor of the crystalline sheaf $\bigwedge^2(\mathbb{H}' \otimes \mathbb{Z}_{p^r})$. So by Lemma 5.2 (i), $(\mathrm{Sym}^2 \tilde{\mathbb{V}}_1(-2^{d-2}))_{x^0}$ is a crystalline lattice for the group $\mathrm{Gal}_{\mathrm{Frac}W(k)}$ and by (iii), one has the equality (after taking the eigen component to the eigenvalue s_0)

$$(\mathbf{E}_0)_x = \mathbf{E}^*(\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2}))_{x^0})_0.$$

Then by Lemma 5.2 (ii) it is to determine the filtration of $D_{crys}(\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2}))_{x^0} \otimes \mathbb{Q}_p)_0$. Consider the $\mathrm{Gal}_{\mathrm{Frac}W(k)}$ -representation $\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2}))_{x^0} \otimes \mathbb{Q}_p$. It is equal to $\mathrm{Sym}^2(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2}) \otimes \mathbb{Q}_p)$, and by Proposition 3.11 $\mathbb{V}_{1,x^0} \otimes \mathbb{Q}_p$ is crystalline for an open subgroup $\mathrm{Gal}_E \subset \mathrm{Gal}_{\mathrm{Frac}W(k)}$. As

$$\mathrm{Sym}^2(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2})) = \mathrm{Sym}^2(\mathbb{V}_{1,x^0}) \otimes_{\mathbb{Z}_{p^r}} \det(\mathbb{V}_{1,\sigma,x^0}) \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} \det(\mathbb{V}_{1,\sigma^{r-1},x^0}),$$

and the functor D_{crys} computes with a Schur functor for a crystalline representation, we have

$$D_{crys}(\mathrm{Sym}^2(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2}) \otimes \mathbb{Q}_p))_0 = \mathrm{Sym}^2(D_{crys}(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2}) \otimes \mathbb{Q}_p)_0),$$

which is seen to be naturally isomorphic to

$$[\mathrm{Sym}^2 M_0 \otimes \bigotimes_{i=1}^{r-1} \det(M_i)] \otimes \mathrm{Frac}(W(k_E)).$$

This shows that the filtration of $(\mathbf{E}_0)_x \otimes \mathrm{Frac}W(k_E)$ is nontrivial. So is the filtration on $(\mathbf{E}_0)_x \otimes \mathrm{Frac}W(k)$. \square

We proceed to construct a weak Hasse-Witt pair $(\mathcal{P}_0, \tilde{F}_{rel}^r)$.

Construction of line bundle.

Consider the filtration on the factor \mathbf{E}_0 . As the Hodge filtration on H is filtered free, the induced filtration on \mathbf{E}_0 by Proposition 5.6 is also filtered free.

Lemma 5.8. *The filtration F on \mathbf{E}_0 is nontrivial with form*

$$\mathbf{E}_0 = F^0 \mathbf{E}_0 \supset F^1 \mathbf{E}_0 \supset F^2 \mathbf{E}_0$$

and each grading is locally free of rank one.

Proof. As it is filtered free, it suffices to show this over a point x as above. Then it follows from Lemma 5.7 and the proofs of Propositions 3.11, 3.17. \square

By the lemma we put $\mathcal{P} = \frac{\mathbf{E}_0}{F^1 \mathbf{E}_0}$ and \mathcal{P}_0 be the modulo p reduction of \mathcal{P} which is defined over M_0 . We obtain a certain understanding of the line bundle \mathcal{P}^0 over M^0 by taking a comparison with the variation of Hodge structures at infinity. Let

$(H_{dR}^1, F_{hod}, \nabla^{GM})$ be the automorphic vector bundle over M_K coming from the universal family of abelian varieties over M_K . One has a natural isomorphism

$$(H, F, \nabla) \otimes_{\mathcal{O}_p} F_p \cong (H_{dR}^1, F_{hod}, \nabla^{GM}) \otimes_F F_p.$$

We intend to show a tensor decomposition of $(H', F, \nabla) \otimes_{\mathcal{O}_{p,\tau}} \bar{\mathbb{Q}}_p$ of the form

$$(H', F, \nabla) \otimes_{\mathcal{O}_{p,\tau}} \bar{\mathbb{Q}}_p = (H_1, F_1, \nabla_1) \otimes \cdots \otimes (H_d, F_d, \nabla_d),$$

among which the F_1 is the unique nontrivial tensor factor. For this we apply the theory of de Rham cycles as explained in §2.2 [18]. Let $\{s_{\alpha,B}\} \subset H_{\mathbb{Q}}^{\otimes}$ be a finite set of tensors defining the subgroup $G_{\mathbb{Q}} \subset \mathrm{GL}(H_{\mathbb{Q}})$. By Corollary 2.2.2 in loc. cit. it defines a set of de Rham cycles $\{s_{\alpha,dR}\} \subset (H_{dR}^1)^{\otimes}$ defined over the reflex field $\tau(F)$, which are by definition ∇^{GM} -parallel and contained in Fil^0 .

Lemma 5.9. *The set of de Rham cycles $\{s_{\alpha,dR}\}$ induces a tensor decomposition*

$$(H_{dR}^1, F_{hod}, \nabla^{GM}) \otimes_{F,\tau} \bar{\mathbb{Q}} = \left[\bigotimes_{i=1}^d (H_{dR,i}^1, F_{hod,i}, \nabla_i^{GM}) \right]^{\oplus 2^{\epsilon(D)}}$$

such that the $F_{hod,1}$ is the unique nontrivial tensor factor.

Proof. Consider the natural projection of complex analytic spaces:

$$\pi : \tilde{M}_{an} := X \times G(\mathbb{A}_f)/K \rightarrow M_K(\mathbb{C})$$

The pull-back of $(H_{dR}^1, \nabla^{GM}) \otimes \mathbb{C}$ over $M_K(\mathbb{C})$ via π is trivialized, and by the de Rham isomorphism it is isomorphic to $(H_{\mathbb{Q} \otimes \mathbb{Q}} \mathcal{O}_{\tilde{M}_{an}}, 1 \otimes d)$. By a similar discussion on the tensor decomposition of the $G(\mathbb{Q})$ -representation $H_{\mathbb{Q} \otimes \mathbb{Q}} \mathbb{C}$ as given in §3.1, the tensors $s_{\alpha,B} \otimes 1$ s induce a tensor decomposition of $\pi^*((H_{dR}^1, \nabla^{GM}) \otimes \mathbb{C})$. It is $G(\mathbb{Q})$ equivariant by construction, and hence descends to a decomposition on $(H_{dR}^1, \nabla^{GM}) \otimes \mathbb{C}$. This is the same tensor decomposition induced by the tensors $s_{\alpha,dR}$ s. Since they are defined over $\tau(F)$, the tensor decomposition already happens over $\bar{\mathbb{Q}}$. We have also to check the property about the filtration in the tensor decomposition. Note that the Hodge filtration $\pi^*(H_{dR}^1, F_{hod}) \otimes \mathbb{C}$ over the point $[0 \times id]$ is induced from $\mu_{h_0} : \mathbb{G}_m(\mathbb{C}) \rightarrow G_{\mathbb{C}} \subset \mathrm{GL}(H_{\mathbb{C}})$. The assertion follows then from the definition of h_0 in §2. \square

Composed with the embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, we obtain the claimed tensor decomposition on $(H', F, \nabla) \otimes_{\mathcal{O}_{p,\tau}} \bar{\mathbb{Q}}_p$. Taking the grading with respect to $F_{hod,i}$, one obtains the associated Higgs bundle (E_i, θ_i) with $(H_{dR,i}^1, F_{hod,i}, \nabla_i^{GM})$. By the lemma, only θ_1 is nontrivial. In fact it is a maximal Higgs field (see [38]), that is,

$$\theta_1 : F_{hod,1}^1 \xrightarrow{\cong} \frac{H_{dR,1}^1}{F_{hod,1}^1} \otimes \Omega_{M_K \otimes \bar{\mathbb{Q}}}.$$

Actually over each connected component of M_K , $\theta_1 \otimes \mathbb{C}$ is a morphism of locally homogenous bundles of rank one. Then it must be an isomorphism, because it will otherwise be zero, and together with the zero Higgs fields on the other factors $E_i, i \geq 2$, this implies that the Kodaira-Spencer map of the universal family is trivial, which is absurd. As both $F_{hod,1}^1$ and $\frac{H_{dR,1}^1}{F_{hod,1}^1}$ are locally homogenous line bundles over each connected component of M_K , their isomorphism classes are

determined by the corresponding representations of $K_{\mathbb{R}} \otimes \mathbb{C}$, where $K_{\mathbb{R}}$ is the stabilizer of $G(\mathbb{R})$ at $0 \in X$. In this way one easily shows that they are dual to each other. By putting $\mathcal{L} := F_{\text{hod},1}^1$, one has then

$$\theta_1 : \mathcal{L} \cong \mathcal{L}^{-1} \otimes \Omega_{M_K \otimes \bar{\mathbb{Q}}}.$$

Remark 5.10. Kisin ([18]) showed further that the de Rham cycles $s_{\alpha,dR}$ extend to sections of (H, F, ∇) (see Corollary 2.3.9 in loc. cit.). This implies that the tensor decomposition is defined over $\bar{\mathbb{Z}}_p$ (i.e. inverting p is not necessary). We do not need this fact in the following. However this will allow us to discuss the properties about the modulo p reduction of the rank two Higgs bundles $\{(E_i, \theta_i)\}_{1 \leq i \leq d}$ (compare §6-§7 [34]).

By abuse of notation, we use \mathcal{L} again to denote the base change of \mathcal{L} to

$$\bar{M}^0 := M^0 \otimes_{F_p} \bar{\mathbb{Q}}_p = (M_K \otimes_{F,\tau} \bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}_p.$$

Lemma 5.11. *Over \bar{M}^0 , one has a natural isomorphism*

$$\mathcal{P}^0 \cong \mathcal{L}^{-2}.$$

Proof. In fact we show that there is a natural isomorphism

$$(\mathbf{E}_0, F) \otimes \mathcal{O}_{\bar{M}^0} \cong \text{Sym}^2(H_1, F_1).$$

We raise the defining field of M^0 so that it contains the defining field of H_i , $1 \leq i \leq d$ and \mathbb{Q}_{p^d} . By abuse of notation, we use the same notation to mean an above object after the base change. Let $U = \text{Spec} R \subset M$ be a small affine subset. We have a natural isomorphism

$$\mathbb{H}' \otimes_{\mathbb{Z}_p} B(R) \cong H' \otimes_R B(R)$$

respecting Γ_R -actions and filtrations (we forget the ϕ s in the isomorphism). As $\mathbb{Q}_{p^d} \subset B(R)$, we can write it as

$$(\tilde{\mathbb{V}}_1 \otimes \cdots \otimes \tilde{\mathbb{V}}_d) \otimes_{\mathbb{Q}_{p^d}} B(R) \cong (H_1 \otimes \cdots \otimes H_d) \otimes_{R[\frac{1}{p}]} B(R),$$

or

$$\bigotimes_{i=1}^d [\tilde{\mathbb{V}}_i \otimes_{\mathbb{Q}_{p^d}} B(R)] \cong \bigotimes_{i=1}^d [H_i \otimes_{R[\frac{1}{p}]} B(R)].$$

In the comparison the tensor factor with numbering is preserved, because it is so over a general $\bar{\mathbb{Q}}$ -rational point of each connected component of U by a result of Blasius and Wintenberger (see [2], see also §4[32]), which asserts that in the p -adic comparison the tensors $s_{\alpha,et}$ and $s_{\alpha,dR}$ correspond. Then taking the second wedge (symmetric) power for n even (odd) of the above isomorphism, we find the isomorphism

$$\text{Sym}^2 \mathbb{V}_1 \otimes_{\mathbb{Q}_{p^d}} B(R) \cong \text{Sym}^2 H_1 \otimes_{R[\frac{1}{p}]} B(R)$$

which respects Γ_R -actions and filtrations. Taking Γ_R -invariants of both sides, we obtain the claimed isomorphism over $\text{Spec} R[\frac{1}{p}]$. By the naturalness of the comparison, the local isomorphisms glue into a global one. \square

By the main theorems of Langton [19] (or using the theory of Picard scheme), the line bundle \mathcal{L}^{-1} extends over $M \otimes \bar{\mathbb{Z}}_p$ with the modulo p reduction \mathcal{L}_0^{-1} , and the isomorphism in the above lemma specializes to an isomorphism between \mathcal{P}_0 and \mathcal{L}_0^{-2} . So we have shown the first isomorphism in the following

Proposition 5.12. *Over \bar{M}_0 , one has natural isomorphisms*

$$\mathcal{P}_0 \cong \mathcal{L}_0^{-2} \cong \Omega_{M_0}^{-1}$$

Proof. We have shown that over \bar{M}^0 the Higgs field θ_1 induces an isomorphism $\mathcal{L}^2 \cong \Omega_{M^0}$. For the same reason as above, this isomorphism specializes into an isomorphism $\mathcal{L}_0^2 \cong \Omega_{M_0}$. \square

Construction of Frobenius.

For each small affine $U \in \mathcal{U}$, we choose a Frobenius lifting $F_U : \hat{U} \rightarrow \hat{U}$ (\hat{U} is the p -adic completion of U). As \mathbf{E}_0 is an object in $MF_{big,r}^\nabla(M)$, there is a map

$$\phi_{r,F_U} : F_U^{*r} \mathbf{E}_0|_{\hat{U}} \rightarrow \mathbf{E}_0|_{\hat{U}}.$$

By Proposition 5.6, ϕ_{r,F_U} is the restriction of the second wedge (symmetric) power of the r -th iterated relative Frobenius map $\phi_{F_U} : F_U^* H'|_{\hat{U}} \rightarrow H'|_{\hat{U}}$ for d even (odd) to the direct factor $\mathbf{E}_0|_{\hat{U}}$.

Lemma 5.13. *For each U , the image $\phi_{r,F_U}(F_U^{*r} \mathbf{E}_0|_{\hat{U}}) \subset \mathbf{E}_0|_{\hat{U}}$ is divisible by p^{r-1} , but not divisible by p^r .*

Proof. In the p -adic filtration

$$\mathbf{E}_0|_{\hat{U}} \supset p\mathbf{E}_0|_{\hat{U}} \cdots \supset p^{i-1}\mathbf{E}_0|_{\hat{U}} \supset p^i\mathbf{E}_0|_{\hat{U}} \supset \cdots,$$

there is a unique i with the property

$$p^{i-1}\mathbf{E}_0|_{\hat{U}} \supset \phi_{r,F_U}(F_U^{*r} \mathbf{E}_0|_{\hat{U}}) \not\supset p^i\mathbf{E}_0|_{\hat{U}}.$$

It is to show that $i = r$, or equivalently that the images of $\phi_{r,F_U}(F_U^{*r} \mathbf{E}_0|_{\hat{U}})$ in the successive gradings $\frac{p^{i-1}\mathbf{E}_0|_{\hat{U}}}{p^i\mathbf{E}_0|_{\hat{U}}}$ are zero for $1 \leq i < r$ and is nonzero for $i = r$. Let $x_0 \in \hat{U}(k)$ and \hat{U}_{x_0} the completion of \hat{U} at x_0 . It is equivalent to show the above statement over each \hat{U}_{x_0} . This follows from the description of the relative Frobenius ϕ over the formal neighborhood \hat{U}_{x_0} as described in §4.2 and the result for the closed point x_0 . In detail it goes as follows: By Lemma 5.7, the filtered ϕ^r -module $\mathbf{E}_0|_x$ is isomorphic to $\text{Sym}^2 M_0 \otimes (\bigotimes_{i=1}^{r-1} \det M_i) \otimes \text{unit crystal over } W(\bar{k})$. By Proposition 3.17, the Newton slope of the rank one ϕ^r -module $\det M_i, i \geq 1$ is either 1×1 or 1×2 . In the former case, the Newton slopes of $\text{Sym}^2 M_0$ are $\{1 \times 0, 1 \times 1, 1 \times 2\}$. These imply that $\phi_r(\mathbf{E}_0|_x)$ is always divisible by p^{r-1} . By Remark 3.19, the former case does occur for a certain x_0 . So $\phi_r(\mathbf{E}_0|_x)$ is not divisible by p^r at such a closed point. \square

As $\phi_{r,F_U}(F_U^{*r} F^1 \mathbf{E}_0|_{\hat{U}}) \subset p^r \mathbf{E}_0|_{\hat{U}}$, the composite of the following morphisms

$$F_U^{*r} F^1 \mathbf{E}_0|_{\hat{U}} \hookrightarrow F_U^{*r} \mathbf{E}_0|_{\hat{U}} \xrightarrow{\frac{\phi_{r,F_U}}{p^{r-1}}} \mathbf{E}_0|_{\hat{U}} \xrightarrow{pr} \mathcal{P}|_{\hat{U}} \xrightarrow{\text{mod } p} \mathcal{P}_0|_{U_0}$$

is zero. As a result it gives the following morphism

$$\frac{F_U^{*r} \mathbf{E}_0|_{\hat{U}}}{F_U^{*r} F^1 \mathbf{E}_0|_{\hat{U}}} = F_U^{*r} \mathcal{P}|_{\hat{U}} \rightarrow \mathcal{P}_0|_{U_0},$$

which clearly factors further through $F_U^{*r} \mathcal{P}|_{\hat{U}} \xrightarrow{\text{mod } p} F_{U_0}^{*r} \mathcal{P}_0|_{U_0}$. Thus we obtain a morphism $F_{U_0}^{*r} \mathcal{P}_0|_{U_0} \rightarrow \mathcal{P}_0|_{U_0}$ which we denote by $[\frac{\phi_{r,F_U}}{p^{r-1}}]$.

Lemma 5.14. *The local maps $\{[\frac{\phi_{r,F_U}}{p^{r-1}}]\}_{U \in \mathcal{U}}$ glue into a global one:*

$$\tilde{F}_{rel}^r : F_{M_0}^{*r} \mathcal{P}_0 \rightarrow \mathcal{P}_0.$$

Proof. It is equivalent to show the following statement: For two different Frobenius liftings F_U and F'_U of the absolute Frobenius F_{U_0} , and for a local section of $(F_{M_0}^{*r} \mathcal{P}_0)(U_0)$ of form $F_{U_0}^{*r} s_0$ with $s_0 \in \mathcal{P}_0(U_0)$, one has the equality

$$[\frac{\phi_{r,F_U}}{p^{r-1}}](F_{U_0}^{*r} s_0) = [\frac{\phi_{r,F'_U}}{p^{r-1}}](F_{U_0}^{*r} s_0).$$

Let s be an element of $\mathbf{E}_0(\hat{U})$ lifting s_0 . It is to show that

$$(\frac{\phi_{r,F'_U}}{p^{r-1}} F_U^{*r} - \frac{\phi_{r,F_U}}{p^{r-1}} F_U^{*r})(s) \in p \mathbf{E}_0(\hat{U})$$

Note that by replacing the Frobenius ϕ_r of \mathbf{E}_0 with $\frac{\phi_r}{p^{r-1}}$ one obtains another object in $MF_{big,r}^\nabla(M)$, which is denoted by \mathbf{E}'_0 . Let $x_0 \in \hat{U}(k)$ and \hat{U}_{x_0} be the completion. Take an isomorphism $\hat{U}_{x_0} \cong W(k)[[t]]$. Then F_U and F'_U restrict to two Frobenius lifting on \hat{U}_{x_0} . Take any local section s' of $\mathbf{E}'_0(\hat{U}_{x_0})$. Then over \hat{U}_{x_0} , one has the Taylor formula (see §7 [13], Theorem 2.3 [12], page 16 [18]): Write $\partial = \partial t$ and $z = F'_U(t) - F_U(t)$

$$\frac{\phi_{r,F'_U}}{p^{r-1}} F_U^{*r}(s') = \sum_{i=0}^{\infty} \frac{\phi_{r,F_U}}{p^{r-1}} F_U^{*r}(\nabla_{\partial}^i(s')) \otimes \frac{z^i}{i!}.$$

Note that as z is divisible by p , $\frac{z^i}{i!}$ is divisible by p for all $i \geq 1$. So the difference $\frac{\phi_{r,F'_U}}{p^{r-1}} F_U^{*r}(s') - \frac{\phi_{r,F_U}}{p^{r-1}} F_U^{*r}(s')$ belongs to $p \mathbf{E}'_0(\hat{U}_{x_0})$. \square

Let $\mathcal{S} \subset M_0(\bar{k})$ be the supersingular locus of $f_0 : X_0 \rightarrow M_0$.

Proposition 5.15. *The morphism \tilde{F}_{rel}^r is nonzero and takes zero at $x_0 \in M_0(\bar{k})$ iff $x_0 \in \mathcal{S}$.*

Proof. The morphism \tilde{F}_{rel}^r is nonzero because of Lemma 5.13. And when and only when it takes zero at x_0 , the Newton slopes of the factors M_i in the proof of Lemma 5.13 take value in $\{2 \times 1\}$, which by the proof of Theorem 3.18 implies that $x_0 \in \mathcal{S}$. \square

Thus the support of the zero locus of \tilde{F}_{rel}^r coincides with the supersingular locus \mathcal{S} . We call the pair $(\mathcal{P}_0, \tilde{F}_{rel}^r)$, that is defined over M_0 , a weak Hasse-Witt pair for the Shimura curve M_K at the prime \mathfrak{p} .

5.4. A mass formula. In this paragraph we deduce a mass formula for the supersingular locus \mathcal{S} from the weak Hasse-Witt pair $(\mathcal{P}_0, \tilde{F}_{rel}^r)$. It will be clear that we shall determine the multiplicity of the Frobenius degeneracy at a supersingular point. To that we have the following result.

Proposition 5.16. *The vanishing order of \tilde{F}_{rel}^r at each supersingular point is two.*

This is a local statement. Take $x_0 \in \mathcal{S} \cap M_0(\bar{k})$. The theory as discussed in §4.1 provides us with a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$ -divisible module B' such that Corollary 4.13 holds. It is also clear that B' is supersingular. In this case, it is a formal p -divisible group. By Lemma 5.7, the above statement can be deduced from the corresponding result for the universal filtered Dieudonné module associated to a versal deformation of a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$ -divisible module. To this end we shall apply the theory of display (see [30],[31],[41]-[42]) for a local expression of the Frobenius. Note that $\text{Sym}^2 \mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)$ is contained as a direct factor in $\bigwedge^2(\bigotimes_{i=0}^r \mathcal{N}_i)$ (resp. $\text{Sym}^2(\bigotimes_{i=0}^r \mathcal{N}_i)$) for r even (resp. odd). The induced Frobenius on the factor $\text{Sym}^2 \mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)$ from the second wedge/symmetric power of ϕ_{ten}^r on $\bigotimes_{i=0}^r \mathcal{N}_i$ is denoted by $\phi_{ten}^r \otimes^2$. We have then the following

Proposition 5.17. *The vanishing order of $\phi_{\mathcal{N}_0} \bmod p$ on $\frac{\mathcal{N}_0}{\text{Fil}^1 \mathcal{N}_0}$ along the equal characteristic deformation at B' is one, and that of $\frac{\phi_{ten}^r \otimes^2}{p^{r-1}} \bmod p$ on $\frac{\text{Sym}^2 \mathcal{N}_0}{\text{Fil}^1 \text{Sym}^2 \mathcal{N}_0} \otimes \bigotimes_{i=1}^{r-1} \det \mathcal{N}_i$ is two.*

Proof. It is possible to write down the universal Dieudonné display of a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$ -divisible module by the work of Zink (see [41]-[42], particularly Example 1.19 [41] for the display for a Drinfel'd module of height one). However to our current purpose it suffices to write down the display over the equal-characteristic deformation (see [30], [31], §2 [15]). For simplicity we shall take $r = 2$ in the following argument. The proof for a general r is completely the same. Let (N, F, V) be the covariant Dieudonné module of the Cartier dual of the Drinfel'd $\mathcal{O}_{\mathfrak{p}}$ -divisible module B' over \bar{k} . So we have the eigen decomposition with respect to the endomorphism $\mathcal{O}_{\mathfrak{p}} \cong \mathbb{Z}_{p^2}$:

$$N = N_0 \oplus N_1.$$

Choose basis $\{X_i, Y_i\}$ for N_i for $i = 0, 1$. To write down the display, we need to arrange the order of the basis elements into $\{Y_0, X_1, Y_1, X_0\}$ with the understanding that X_0 modulo p is the basis element of $\frac{VN}{pN}$ which is one dimensional \bar{k} -vector space. Then the display under the chosen basis is given by the matrix:

$$\begin{pmatrix} A_{3 \times 3} & B_{3 \times 1} \\ C_{1 \times 3} & D_{1 \times 1} \end{pmatrix} = \begin{pmatrix} 0 & c_1 & d_1 & 0 \\ b_1 & 0 & 0 & a_1 \\ b_2 & 0 & 0 & a_2 \\ 0 & c_2 & d_2 & 0 \end{pmatrix}$$

This is an invertible matrix, i.e.

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$$

is a unit. Since both determinants are elements in $W(\bar{k})$, it implies that each determinant is a unit in $W(\bar{k})$. The universal equal-characteristic deformation ring of B' as p -divisible group is $\bar{k}[[t_0, t_1, t_2]]$. Let $T_i \in W(\bar{k}[[t_0, t_1, t_2]])$, $0 \leq i \leq 2$ be the Teichmüller lifting of t_i . Then by Norman, Norman-Oort, the display over the universal equal-characteristic deformation is given by

$$\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix},$$

where $T = \begin{pmatrix} T_0 \\ T_1 \\ T_2 \end{pmatrix}$. And the Frobenius on the universal display is given by

$$M_1 := \begin{pmatrix} A + TC & p(B + TD) \\ C & pD \end{pmatrix}.$$

We need to determine the one dimensional sublocus of $\mathrm{Spf}(\bar{k}[[t_0, t_1, t_2]])$ where B' deforms as a Drinfel'd module. Take $s \in \mathbb{Z}_{p^2}$ to be a primitive element. Then the endomorphism of N given by s has the matrix form (using the same basis):

$$M_2 := \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^\sigma & 0 & 0 \\ 0 & 0 & \xi^\sigma & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix}.$$

The universal display of the Drinfel'd module has the property that the endomorphism matrix commutes with the Frobenius. That is one has

$$M_1 M_2^\sigma = M_2 M_1.$$

Now by an easy computation one finds that the one dimensional deformation as the Drinfel'd module is given by $t_1 = t_2 = 0$. Write $t = t_0$. Thus the two-iterated Frobenius $\phi_{\mathcal{N}_{B'}}^2$ on $\mathcal{N}_{B'}$ along the equal-characteristic deformation is displayed by

$$\phi_{\mathcal{N}_{B'}}^2 \{Y_0, X_1, Y_1, X_0\} = \{Y_0, X_1, Y_1, X_0\} \Phi,$$

where $\Phi = M_1 M_1^\sigma$ is equal to

$$\begin{pmatrix} \Phi_{11} & 0 & 0 & \Phi_{14} \\ 0 & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{33} & 0 \\ \Phi_{41} & 0 & 0 & \Phi_{44} \end{pmatrix}.$$

The nontrivial entries are given by

$$\begin{aligned} \Phi_{11} &= (b_1^\sigma c_1 + b_2^\sigma d_1) + (b_1^\sigma c_2 + b_2^\sigma d_2)t, & \Phi_{14} &= (pa_1^\sigma c_1 + pa_2^\sigma d_1) + (pa_1^\sigma c_2 + pa_2^\sigma d_2)t, \\ \Phi_{22} &= (b_1 c_1^\sigma + pa_1 c_2^\sigma) + b_1 c_2^\sigma t^\sigma, & \Phi_{23} &= (b_1 d_1^\sigma + pa_1 d_2^\sigma) + b_1 d_2^\sigma t^\sigma, \\ \Phi_{32} &= (b_2 c_1^\sigma + pa_2 c_2^\sigma) + b_2 c_2^\sigma t^\sigma, & \Phi_{33} &= (b_2 d_1^\sigma + pa_2 d_2^\sigma) + b_2 d_2^\sigma t^\sigma, \\ \Phi_{41} &= b_1^\sigma c_2 + b_2^\sigma d_2, & \Phi_{44} &= pa_1^\sigma c_2 + pa_2^\sigma d_2. \end{aligned}$$

Consider first the element Φ_{11} : Modulo p , it is equal to the iterated Hasse-Witt map on $\frac{\mathcal{N}_0}{F\ell^1 \mathcal{N}_0}$. As we require that B' lies in the supersingular locus which is a

finite set, it follows that

$$b_1^\sigma c_1 + b_2^\sigma d_1 = 0 \pmod{p}, b_1^\sigma c_2 + b_2^\sigma d_2 \neq 0 \pmod{p}.$$

This shows the first assertion in the statement. So we can write that $b_1^\sigma c_1 + b_2^\sigma d_1 = pv_1$, $b_1^\sigma c_2 + b_2^\sigma d_2 = u_1$, where u_1 is a unit. Consider the induced Frobenius on $\mathrm{Sym}^2 \mathcal{N}_0 \otimes \bigwedge^2 \mathcal{N}_1$. We shall compute the coefficient before the element $Y_0^2 \otimes X_1 \wedge Y_1$, which is the basis element of $\frac{\mathrm{Sym}^2 \mathcal{N}_0}{\mathrm{Fil}^1 \mathrm{Sym}^2 \mathcal{N}_0} \otimes \det \mathcal{N}_1$, under the map $\frac{\phi_{\mathrm{ten}}^2}{p} \pmod{p}$. Using the above matrix expression of $\phi_{\mathcal{N}_{B'}}^2$, one computes that the local expression is given by

$$[-u_1^2 \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}^\sigma] t^2 + v_2 t^p + v_3 t^{p^2}.$$

By the previous discussion we know that the coefficient before t^2 is a unit. As p is assumed to be odd, it follows that the multiplicity is equal to two. \square

Now the proof of Proposition 5.16 is clear:

Proof. By the construction of \tilde{F}_{rel}^r , its vanishing order at x_0 is equal to that of $\frac{\phi_r}{p^{r-1}} \pmod{p}$ on $\frac{\mathbf{E}_0}{F^1 \mathbf{E}_0}$ along \hat{M}_{0,x_0} . Note that the closed formal subscheme $\hat{M}_{0,x_0} \subset \hat{M}_{x_0}$ represents the equal-characteristic deformation direction. By Lemma 5.7, the restriction of \mathbf{E}_0 to \hat{M}_{x_0} is naturally isomorphic to $\hat{\xi}_{crys}[\mathrm{Sym}^2 \mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)]$. Thus the result follows from Proposition 5.17. \square

Corollary 5.18. *Let \mathcal{S} be the supersingular locus of $f_0 : X_0 \rightarrow M_0$. Then in the Chow ring of \bar{M}_0 one has the cycle formula*

$$2\mathcal{S} = (1 - p^r)c_1(M_0),$$

where $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$. Consequently one has the following mass formula

$$|\mathcal{S}| = (p^r - 1)(g - 1),$$

where g is the genus of the Shimura curve M_K .

Proof. By Proposition 5.15 and Corollary 5.16 it follows that

$$2\mathcal{S} = (p^r - 1)c_1(\mathcal{P}_0).$$

By Proposition 5.12, one has further

$$c_1(\mathcal{P}_0) = -2c_1(\mathcal{L}_0) = -c_1(M_0).$$

By taking the degree of the cycle formula, one obtains the mass formula as above. \square

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